CONTINUUM MECHANICS AND BIOMECHANICS
(Mécanique des milieux continus et biomécanique)

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1. INTRODUCTION: SOME GENERAL TRAITS

Biomechanics may be defined as mechanics applied to biological systems (from bios = life, méchaniké = mechanics). It deals with the mechanical properties of bio-materials. It studies the mechanical principles of the state and evolution of living organisms, in particular, their motion, structure, and reactions to mechanical loads. If a continuum standpoint is readily adopted, then the whole of continuum mechanics may be implied, whether with more or less flowing fluids (*rheology*) or deformable solids. Here we put the emphasis on the latter case with the possible occurrence of finite deformations and visco-elasticity (in soft tissues and bones).

The a priori complexity of the subject matter must be emphasized from the outset as the spatial-temporal evolution of biological materials in response to a system of (internally or externally) imposed forces cannot be a simple affair, for the object of study generally is a system with a hierarchical structure with multi-components in reciprocal interaction, and with the frequent possibility of interaction with the exterior of the system. These three points underline (i) the possibly necessary consideration of a microstructure and the related problem of the choice of a pertinent level of description, (ii) the possible introduction of a theory of mixtures, and (iii) the eventual consideration of open thermodynamic systems, what is seldom envisaged in inert media, the usual objet of study of mechanical engineers. In this last framework, we may have to contemplate seriously a possible non-conservation of mass and the typical phenomenon of growth or resorption, what really provides the quintessence of the time development of living matter, although this may also look very much like some phenomena of phase transformation in inert bodies.

From these typical traits there follow the inescapable complexity of the subject matter and its inherent multi-disciplinary features, hence the required sophistication of the continuum vision and of its associated mechanics, thermo-mechanics in a more inclusive context. To illustrate this very point, we can notice the frequent occurrence of instantaneous deformations followed by retarded (visco-elastic) deformations as also a hysteretic behaviour. Such visco-elastic media exhibit a mechanical response that may depend on the magnitude of the considered deformation, on the velocity of application of external forcing and of their duration, and of the whole history of the past deformation. There remains the essential choice of whether small or finite deformations should be considered although it is matter-of-factly evident that many biological media (in particular soft tissues) are ideal candidates for a finite-strain description. It is also obvious that many such materials present a typical response that is quite different from that of inert matter, with a possible locking in strain and an accompanying hysteresis such as shown in Figure 1.

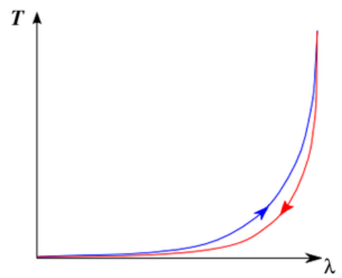

Figure 1. Graph of Lagrangian traction (T) versus stretch ratio (λ) of a preconditioned soft tissue.

Technical details and formulas are given in the oral session at the MECAMAT School held in Aussois, January 2016.

2. SPECIFIC DOMAINS OF INTEREST

Here we must distinguish between *soft* and *hard* tissues, our own domains of interest. We count among *soft tissues* tendons, ligaments, skin, fibrous tissues, muscles, cartilage, and blood vessels. Their essential constitutive elements are connective tissues, collagen (macromolecules, oriented fibres), elastin (long flexible molecules). In particular, the role of collagen must be underlined as it provides the mechanical resistance of tissues to elongation with a noted *anisotropy*. Generally speaking, these soft tissues are more than often composite materials reinforced by fibres (often two networks of such fibres). The mechanical treatment requires the consideration of *finite deformations* and the possible introduction of *hyperelasticity* and pseudo-elasticity. *Residual stresses* play an important role (as demonstrated in the cut of a section, and subsequent opening, of an artery). The phenomena of *growth* (in mechano-biology) and *remodelling* (re-construction) are paramount.

In contrast, *hard* tissues illustrated by wood, shells and bones, especially bone tissues (with osteocytes) with different characterizations (spongious and compact tissues), require only the consideration of small elastic deformations (*linear* elasticity) with an eventual marked influence of some *anisotropy*.

Within our specialized field of interest two examples of complex problems can be cited for the sake of illustration. One concerns the growth of long bones in mammals. The other one deals with growth with the possible occurrence of lines and points of singularity. In the first case, one considers the ossification or slow thermodynamically irreversible evolution of the so-called growth – or epiphyseal - plate, a microstructured transition zone of timewise decreasing thickness between a region of condrocytes (cells of cartilage) and spongy bones at the end of extending long bones. This problem involves the theory of configurational forces (i.e., driving forces; cf. Maugin, 2011) in hyperlasticity and was examined by the author in collaboration with mechanicians of phase transformations at St Petersburg – team of A.B. Freidin. The second complex problem is that which examines the extension of tissues when discontinuity lines appear in the process with the ultimate formation of singular points (e.g., apices). This may be the growth of an animal’s horn (case of rhinoceros or antilopes) or the natural growth of some seashells (e.g., *sinum cymba* or *turitella communis*). The appropriate theory was recently developed by Ciarletta *et al* (2012, 2013). This requires the consideration of a second-gradient theory of finite strains in thermodynamically open systems with exchanges between components. No need to emphasize the complexity of the resulting approach that calls for the latest developments in the theory of anelasticity and configurational forces.

3. USEFUL STANDARD MODELS OF CONTINUA

These are the most classical models essentially developed in the nineteenth century, namely, linear elasticity in small strains with a possibility of anisotropy provided by networks of reinforcing fibres (cf. Spencer, 1972), and linear viscous fluids that owe so much to the pioneering physiological studies of Poiseuille on viscosity and the theoretical models of Navier, Saint-Venant and Stokes. Linear models of non-Newtonian fluids in the manner of Kelvin-Voigt, Maxwell and others (typical rheological models) may also be of interest. This is documented in most courses and books on continuum mechanics at the undergraduate level.

4. MORE COMPLEX BEHAVIOURS, FINITE-STRAIN ELASTICITY IN PARTICULAR

This requires the consideration of finite-strains and of the whole paraphernalia of nonlinear continuum mechanics. The scheme of this mechanics is usually presented by means of a diagram such as in Figure 2, where one distinguishes between the reference configuration

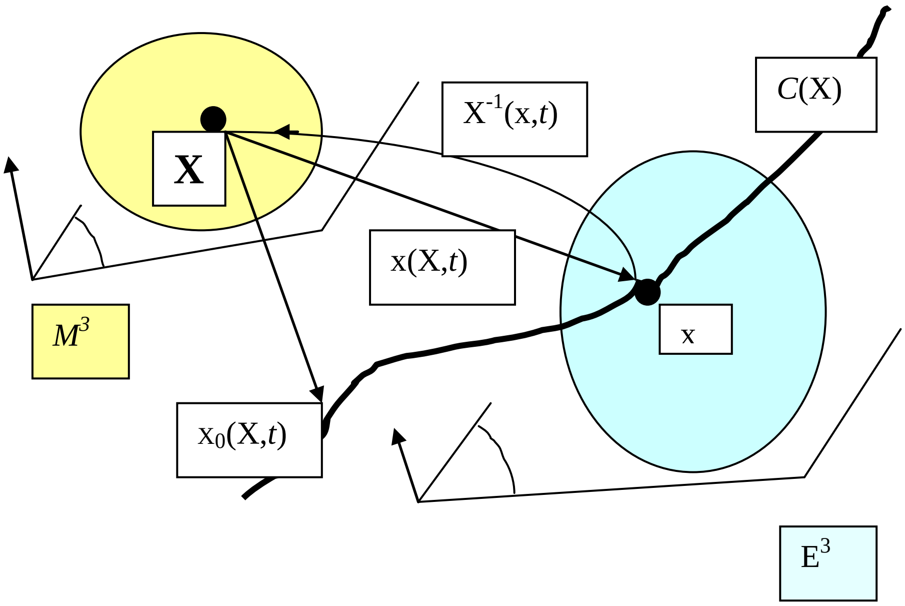

Figure 2. General deformation mapping

K_R and the actual configuration K_t at time t. Classical definitions of the deformation gradient, its inverse, the Jacobian determinant, absolute and relative measures of strains, polar decomposition and principal stretches are given by formulas:

$$\mathbf{x} = \chi(\mathbf{X},\ t),\ \ \mathbf{X} = \chi^{-1}(\mathbf{x}, t) \tag{4.1}$$

$$\mathbf{F} := \nabla_R \chi \equiv \frac{\partial \chi}{\partial \mathbf{X}} \quad , \quad \mathbf{F}^{-1} := \nabla \chi^{-1} \equiv \frac{\partial \chi^{-1}}{\partial \mathbf{x}},\ J_F = \det \mathbf{F} > 0\,, \tag{4.2}$$

$$\mathbf{C}(\mathbf{X}, t) := \mathbf{F}^T \mathbf{F} = \left\{ C_{KL} = F^i_K\, \delta_{ij} F^j_L \right\} \quad , \tag{4.3}$$

$$\mathbf{C}^{-1} := \left(\mathbf{F}^{-1}\right)\left(\mathbf{F}^{-1}\right)^T = \left\{ \left(\mathbf{C}^{-1}\right)^{KL} = \left(\mathbf{F}^{-1}\right)^K_i \delta^{ij} \left(\mathbf{F}^{-1}\right)^L_j \right\} \tag{4.4}$$

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{C} - \mathbf{1}_R\right) \tag{4.5}$$

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R},\ \mathbf{U} = \mathbf{U}^T,\ \mathbf{V} = \mathbf{V}^T,\ \mathbf{R}^T = \mathbf{R}^{-1}, \det \mathbf{R} = +1, \tag{4.6}$$

Principal stretches λ_α , $\alpha = 1,2,3$ are the eigenvalues of the positive definite $\mathbf{U}$ *or* $\mathbf{V}$.
Displacement field is given by

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X},\ \ or\ \overline{\mathbf{u}}(\mathbf{x}, t) \tag{4.7}$$

$$\mathbf{F} = \mathbf{1} + \nabla_R \mathbf{u} \quad ; \quad \mathbf{F}^{-1} = \mathbf{1} - \nabla \overline{\mathbf{u}} \tag{4.8}$$

Of great importance is the operation of pull back and Piola transformation:

$$\overline{\mathbf{A}} = J_F \mathbf{F}^{-1} \mathbf{A} = \left\{ \overline{A}^K = J_F \left(\mathbf{F}^{-1}\right)^{Ki} A_i \right\} \tag{4.9}$$

Change of volume:

$$dv = J_F\, dV\ . \tag{4.10}$$

Kinematics:
Velocity:

$$\mathbf{v}(\mathbf{X},t) := \left.\frac{\partial \chi}{\partial t}\right|_X \quad , \quad \left.\frac{\partial}{\partial t}\right|_{X\ fixed} = \left.\frac{\partial}{\partial t}\right|_{x\ fixed} + \mathbf{v}.\nabla \,. \tag{4.11}$$

Rate of strain :

$$\mathbf{L} = (\nabla \mathbf{v})^T = \dot{\mathbf{F}}\,\mathbf{F}^{-1}, \tag{4.12}$$

$$\mathbf{D} = \mathbf{L}_S = \frac{1}{2}\left((\nabla \mathbf{v})^T + \nabla \mathbf{v}\right) = \mathbf{F}^{-T}\dot{\mathbf{E}}\,\mathbf{F}^{-1} \tag{4.13}$$

Standard balance equations in the Eulerian format:

$$\dot{\rho} + \rho(\nabla.\mathbf{v}) = \left.\frac{\partial \rho}{\partial t}\right|_x + \nabla.(\rho \mathbf{v}) = 0 \quad ; \tag{4.14}$$

$$\rho \dot{\mathbf{v}} - div\ \mathbf{t} = \rho \mathbf{f} \,, \tag{4.15}$$

$$\mathbf{t} = \mathbf{t}^T \quad i.e., \quad t_{ij} = t_{ji} \quad ou \quad t_{[ij]} = 0 \tag{4.16}$$

with (Cauchy’s tetrahedron argument)

$$\mathbf{t}_{(n)}(\mathbf{x},t;\mathbf{n}) = \mathbf{n}.\mathbf{t} \quad . \tag{4.17}$$

Standard balance equations in the Piola-Kirchhoff format:

$$\left.\frac{\partial}{\partial t}\rho_0\right|_X = 0, \quad \rho_0 = \rho\, J_F \,; \tag{4.18}$$

$$\left.\frac{\partial}{\partial t}\mathbf{p}_R\right|_X - div_R \mathbf{T} = \rho_0 \mathbf{f} \,; \tag{4.19}$$

$$\mathbf{F}\mathbf{T} = \mathbf{T}^T\mathbf{F}^T \,; \tag{4.20}$$

with $\mathbf{T}$ the first Piola-Kirchhoff stress.

Thermodynamics of pure finite-strain elasticity (Hyperelastic materials, G. Green) (with W: energy of deformation per unit undeformed volume and ψ : per unit mass, **S**: second (symmetric fully material) Piola-Kirchhoff stress):

$$\mathbf{S} = \mathbf{T}\mathbf{F} = J_F \mathbf{F}^{-1}\mathbf{t}\,\mathbf{F}^{-T} = \left\{ S^{KL} = J_F \left(\mathbf{F}^{-1}\right)^K_{.i} t^{ij} \left(\mathbf{F}^{-1}\right)^L_{.j} \right\} \quad , \quad \mathbf{t} = J_F^{-1}\mathbf{F}\mathbf{S}\,\mathbf{F}^T \tag{4.21}$$

$$\mathbf{S} = \left.\frac{\partial \hat{W}}{\partial \mathbf{E}}\right|_\theta \quad , \quad \mathbf{t} = J_F^{-1}\,\mathbf{F}\left.\frac{\partial \hat{W}}{\partial \mathbf{E}}\right|_\theta \mathbf{F}^T = 2\rho\mathbf{F}\frac{\partial \hat{\psi}}{\partial \mathbf{C}}\mathbf{F}^T \,. \tag{4.22}$$

In presence of material inhomogeneities:

$$\rho_0 = \bar{\rho}_0(\mathbf{X}), \quad W = \hat{W}(\mathbf{E};\mathbf{X}) \ or\ \overline{W}(\mathbf{C};\mathbf{X}), \quad \psi = \hat{\psi}(\mathbf{C};\mathbf{X}). \tag{4.23}$$

Examples:

With the following definitions for strain invariants:

$$I_1 = tr\ \mathbf{C},\ I_2 = \frac{1}{2}\left[(tr\,\mathbf{C})^2 - tr\left(\mathbf{C}^2\right)\right],\ I_3 = \det\ \mathbf{C}, \tag{4.24}$$

we have at hand some celebrated models of isotropic continua, namely

Neo-Hookean material (Rivlin, 1948)

$$W = C(I_1 - 3),\ \ C > 0; \tag{4.25}$$

Money-Rivlin material:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{4.26}$$

or in terms of principal stretches:

$$W = W(\lambda_\alpha) = W(\lambda_1, \lambda_2, \lambda_3) \tag{4.27}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\ I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\ ,\ I_3 = \lambda_1^2\,\lambda_2^2\,\lambda_3^2. \tag{4.28}$$

Biot's constitutive equations:

$$f_\alpha = \frac{\partial W}{\partial \lambda_\alpha}\ ,\ \alpha = 1,2,3. \tag{4.29}$$

Anisotropic media (e.g., Holzapfel with two directions of fibres $\mathbf{a}_1 and\, \mathbf{a}_2$) for which

$$W = U(J;\mathbf{X}) + \overline{W}\left(\overline{\mathbf{C}}, \mathbf{A}_1, \mathbf{A}_2; \mathbf{X}\right),\ \mathbf{A}_1 := \mathbf{a}_{01} \otimes \mathbf{a}_{01}\ , \mathbf{A}_2 := \mathbf{a}_{02} \otimes \mathbf{a}_{02} \tag{4.30}$$

with a distinction between volumetric and distortion behaviours, but with the following reduction in case of incompressibility:

$$\overline{W} = \overline{W}_{iso}\left(\bar{I}_1; \mathbf{X}\right) + \overline{W}_{aniso}\left(\bar{I}_4, \bar{I}_6; \mathbf{X}\right)\ ,\ \bar{I}_4\left(\overline{\mathbf{C}}, \mathbf{a}_{01}\right) = \overline{\mathbf{C}} : \mathbf{A}_1\ ,\ \bar{I}_6\left(\overline{\mathbf{C}}, \mathbf{a}_{02}\right) = \overline{\mathbf{C}} : \mathbf{A}_2, \tag{4.31}$$

where

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}\ ,\ \ J = \det \mathbf{F},\ \overline{\mathbf{F}} = J^{-1/3}\,\mathbf{F},\ \overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}} = J^{-2/3}\mathbf{C}. \tag{4.32}$$

Elastic bio-material of Fung (Fung et al, 1979) for preconditioned tissues:

$$W = \frac{1}{2}\left[q + c(\exp Q - 1)\right] \tag{4.33}$$

with

$$q = C_{KLMN}E_{KL}E_{MN},\ \ Q = B_{KLMN}E_{KL}E_{MN}\ ,\ \ E_{KL} = \frac{1}{2}\left(C_{kl} - \delta_{KL}\right) \tag{4.34}$$

or in terms of principal stretches (isotropic case)

$$W = \frac{1}{2}\left[a\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + b\left(\exp c\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - 1\right)\right]. \tag{4.35}$$

Ogden's model [R. W. Ogden (1972)]. This is given n terms of principal stretches by the expression

$$W = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right). \tag{4.36}$$

Model coupling the neo-Hookean behaviour and Fung's idea: See Holzapfel *et al* (2000. This is considered now as the "*gold standard*" of strain energies for fibre-reinforced biological soft tissues.

5. NONLINEAR DISSIPATIVE BEHAVIOURS

A modern way to deal with nonlinear dissipative mechanical behaviousrs (such as exemplified by plasticity in finite strains) is to introduce a so-called *multiplicative decomposition* of the deformation gradient **F** (following a pioneering proposal by Bilby, Kroener and Lee) and to envisage the exploitation of the thermodynamics with *internal variables of state* (as lengthily illustrated in a book by the author, Maugin, 1999). On the one and we recall the power dissipated by internal stresses in the reference configuration:

$$p^R_{(i)} = tr\left(\mathbf{T}.(\nabla_R \mathbf{v})^T\right) = tr\left(\mathbf{S}.\dot{\mathbf{E}}\right) . \tag{5.1}$$

If α designates the set of internal variables of state (that have for essential property to "dissipate") and are not directly acted upon by external forces, the typicalalt we have top envisage a free energy per unit reference volume in the functional form:

$$W = \overline{W}(\mathbf{F},\theta,\alpha) \ , \ A = -\frac{\partial \overline{W}}{\partial \alpha} , \tag{5.2}$$

where A is the thermodynamic force associated with α , while the *intrinsic dissipation* will read as

$$p_{intr} = A.\dot{\alpha} . \tag{5.3}$$

Indeed exploitation of the Clausius-Duhem inequality yields:

$$\mathbf{T} = \frac{\partial \overline{W}}{\partial \mathbf{F}} , \quad S = -\frac{\partial \overline{W}}{\partial \theta} , \ A\dot{\alpha} \geq 0 , \ (\mathbf{Q}/\theta) . \nabla_R \theta \leq 0 , \tag{5.4}$$

where $\dot{\alpha}$ is principally determined by its conjugate force (Kestin, Rice, 1970s), i.e.,

$$\dot{\alpha} = \overline{\alpha}(A;\mathbf{F},\theta,\alpha) . \tag{5.5}$$

The most popular application of this formalism is provided by *finite-strain plasticity* where one considers the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p , \tag{5.6}$$

where the plasticity « gradient » $\mathbf{F}^p$ is defined from an intermediate or elastically released configuration $K_i = K_{relax}$ defined from K_R . $\mathbf{F}^p$ itself is conceived as an internal variable of state, so that (4.5) yields

$$W = \overline{W}\left(\mathbf{F}^e , \ \alpha = \mathbf{F}^p , \ \theta\right). \tag{5.7}$$

The rate of plastic deformation is then given by

$$\mathbf{D}^p = \left\{\mathbf{F}^e.\mathbf{L}^p.(\mathbf{F}^e)^{-T}\right\}_S = (\mathbf{D}^p)^T \ in \ K_t \ , \mathbf{L}^p = \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} , \tag{5.8}$$

and

$$\mathbf{D}^p_{relax} = \left\{\mathbf{C}^e.\mathbf{L}^p\right\}_S = (\mathbf{D}^p_{relax})^T \ in \ K_{relax} . \tag{5.9}$$

From this formulation one deduces plausible rate equations for the plasticity phenomenon (independent of the rate of application of forces, i.e., without time scale; λ : so-called plastic multiplier) such as

$$\mathbf{D}^p_{relax} = \dot{\lambda} \ \frac{\partial f}{\partial \mathbf{S}_{relax}} \tag{5.10}$$

with second Piola-Kirchhoff stress (in $K_i = K_{relax}$) and S the corresponding entropy density,

$$\mathbf{S}_{relax}=\frac{\partial\widetilde{W}}{\partial\mathbf{E}^{e}}\quad,\qquad S_{relax}=-\frac{\partial\widetilde{W}}{\partial\theta}. \tag{5.11}$$

and a plastic flow surface $f(\mathbf{S}_{relax})=0$ delimiting a convex set C in space spanned by stresses $(\mathbf{S}_{relax})$. This can be generalized for media exhibiting various kinds of hardening by enlarging the space of $\alpha's$. We note that the intrinsic dissipation is given by

$$tr(\mathbf{S}_{relax}.\mathbf{D}^{p}_{relax})=tr(\mathbf{M}_{relax}.\mathbf{L}^{p}),\ \mathbf{M}_{relax}:=\mathbf{S}_{relax}.\mathbf{C}^{e}, \tag{5.12}$$

so that the Mandel stress $\mathbf{M}_{relax}$ is the motive power of plasticity, while $\mathbf{C}^{e}$ or $\mathbf{E}^{e}$ is the observable strain variable (cf. Maugin, 1994).

6. VOLUMETRIC GROWTH

In the presence of volumetric growth the mass is no longer conserved and the continuity equation in K_R should read, including both bulk source and a possible diffusion (where vector **M** is not to be confused with the Mandel stress),

$$\left.\frac{\partial\rho_0}{\partial t}\right|_{\mathbf{X}}=\Pi+\nabla_R.\mathbf{M}. \tag{6.1}$$

On setting

$$\pi=\Pi\ J_F^{-1},\ \ \mathbf{m}=J_F^{-1}\mathbf{F}.\mathbf{M}, \tag{6.2}$$

one obtains the «continuity» equation in K_t in the following form

$$\dot{\rho}+\rho\nabla.\mathbf{v}=\pi+\nabla.\mathbf{m}. \tag{6.3}$$

The balance of linear momentum for the whole body is then proposed as

$$\frac{d}{dt}\int_B\rho\mathbf{v}dv=\int_B\rho\mathbf{f}\,dv+\int_{\partial B}\mathbf{n}.\mathbf{t}da+\int_B(\pi v+\overline{\mathbf{p}})dv, \tag{6.4}$$

localizing as

$$\rho\dot{\mathbf{v}}=\rho\mathbf{f}+\pi\mathbf{v}+\overline{\mathbf{p}}+div\,\mathbf{t}. \tag{6.5}$$

Here $\overline{\mathbf{p}}$ is possibly an irreversible «linear momentum» due to the entrance of new material.
For the sake of simplicity (Epstein and Maugin, 2000) we suppose that $\mathbf{M}=\mathbf{0}$ *or* $\mathbf{m}=\mathbf{0}$ (no diffusion) and $\overline{\mathbf{p}}=\mathbf{0}$, so that equation (6.1) reduces to

$$\left.\frac{\partial\rho_0}{\partial t}\right|_{\mathbf{X}}=\Pi(\mathbf{X},t). \tag{6.6}$$

while (6.5) yields the following form in the Piola-Kirchhoff format:

$$\rho_0\left.\frac{\partial\mathbf{v}}{\partial t}\right|_{X}=\rho_0\mathbf{f}+\Pi\mathbf{v}+div_R\mathbf{T}. \tag{6.7}$$

With the same working hypotheses, the local internal energy equation and the Clausius-Duhem inequality read

$$\left.\frac{\partial}{\partial t}E\right|_{\mathbf{X}}=tr\left(\mathbf{T}.(\nabla_R\mathbf{v})^{T}\right)+\frac{\Pi}{\rho_0}E+\rho_0h_0-\nabla_R.\mathbf{Q}, \tag{6.8}$$

and ($W=E-\theta S$)

$$-\left(\dot{W}+S\dot{\theta}\right)+tr\left(\mathbf{T}.\dot{\mathbf{F}}\right)+\frac{\Pi}{\rho_0}W-\theta^{-1}\mathbf{Q}.\nabla_R\theta\geq 0\,. \tag{6.9}$$

For a thermoelastic material one considers that

$$W=\overline{W}\left(\mathbf{F},\theta;\mathbf{X},t\right)=\rho_0\left(\mathbf{X},t\right)\overline{\psi}\left(\mathbf{F},\theta;\mathbf{X}\right). \tag{6.10}$$

whence the constitutive equations

$$\mathbf{T}=\rho_0\frac{\partial\overline{\psi}}{\partial\mathbf{F}},\ \eta=-\frac{\partial\overline{\psi}}{\partial\theta}\,. \tag{6.11}$$

First-order volumetric growth (Epstein et Maugin, 2000):

We introduce the notion of a reference crystal (configuration noted K_c with respect to which the material behaves elastically with internal energy, free energy and entropy noted E_c, W_c and S_c.) We call "linear transplant" $\mathbf{K}(\mathbf{X})$ the deformation such that $\mathbf{F}_c=\mathbf{F}\mathbf{K}$. The growth «deformation gradient » (not a true gradient) $\mathbf{F}_g$ is such that $\mathbf{K}=\mathbf{F}_g^{-1}$ so that we have a *multiplicative decomposition* of the anelastic type $\mathbf{F}=\mathbf{F}_e\mathbf{F}_g$. The associated deformation rate is defined as

$$\mathbf{L}_K:=\dot{\mathbf{K}}.\mathbf{K}^{-1}=\left.\frac{\partial}{\partial t}\left(\mathbf{F}_g^{-1}\right)\right|_{\mathbf{X}}.\mathbf{F}_g\,. \tag{6.12}$$

As ρ_c is fixed, we immediately obtain that

$$\left.\frac{\partial}{\partial t}\rho_0\right|_{\mathbf{X}}=\rho_c\left(\frac{\partial}{\partial t}J_K^{-1}\right)=-\ \rho_0\ \ tr\ \mathbf{L}_K=\Pi\,. \tag{6.13}$$

Accordingly, in this simplified model Π is known once the kinematics of **K** is known. Growth corresponds to $J_K=\ det\,\mathbf{K}<0$, or $tr\,\mathbf{L}_K<0$. With an energy of deformation given by

$$\overline{W}\left(\mathbf{F},\theta,\mathbf{K};\mathbf{X},t\right)\ =\ J_K^{-1}\,W_c\left(\mathbf{F}\mathbf{K}\left(\mathbf{X},t\right),\,\theta\right)=J_K^{-1}W_c\left(\mathbf{F}_e\,,\theta\right), \tag{6.14}$$

One shows that the residual dissipation inequality reads

$$\Phi=tr\left(\mathbf{M}.\mathbf{L}_K\right)-\theta^{-1}\mathbf{Q}.\nabla_R\theta\geq 0, \tag{6.15}$$

where $\mathbf{M}=\ \mathbf{T}.\mathbf{F}$ now is the *Mandel stress tensor***.** The theory is closed by a relationship between **M** and $\mathbf{L}_K$**.** This will be of the *nonlinear visco-elastic type* [Cf. Takamizawa and Matsuda (1983), Rodriguez *et al* (1994), Taber (1995)]. Following an idea of Cowin (1996), a possible visco-elastic evolution linking **M** and **K** can be written implicitly as

$$\varphi\left(\mathbf{K}\,,\ \ \dot{\mathbf{K}},\ \ \mathbf{M},\ \mathbf{F},\ \dot{\mathbf{F}}\ \ \right)\ =\ 0\ . \tag{6.16}$$

An objective version of this is exemplified by the relation

$$\tilde{\varphi}\left(\tilde{\mathbf{L}}_K\ ,\ \mathbf{M}_0,\,\mathbf{C}_e\ \ ,\ \ \dot{\mathbf{C}}_e\right)=\ 0\,, \tag{6.17}$$

where

$$\mathbf{C}_e=\mathbf{F}_e^T.\mathbf{F}_e\quad,\quad\tilde{\mathbf{L}}_K:=\mathbf{K}^{-1}.\dot{\mathbf{K}}\quad,\quad\mathbf{M}_0=J_K\ \mathbf{K}^{-1}\mathbf{M}\mathbf{K},\qquad\dot{\mathbf{C}}_e=\left.\frac{\partial}{\partial t}\mathbf{C}_e\right|_{\mathbf{X}}\quad. \tag{6.18}$$

One can also use the following representations

$$\tilde{\mathbf{L}}_K\equiv\tilde{\mathbf{L}}_g:=\ \mathbf{F}_g.\dot{\mathbf{F}}_g^{-1}\qquad,\qquad\mathbf{M}_0=\ J_{F_e}^{-1}\mathbf{F}_e.\mathbf{T}.\mathbf{F}_e\,. \tag{6.19}$$

A simple particular case is

$$\tilde{\mathbf{L}}_g = \mathbf{F}(\mathbf{M}_0). \tag{6.20}$$

For isotropic bodies, we shall have

$$\mathbf{S}_e = 2\partial W_c / \partial \mathbf{C}_e \quad \textit{hence} \quad \mathbf{S}_e = \phi_0 \mathbf{1} + \phi_1 \mathbf{C}_e + \phi_2 \mathbf{C}_e^2 \tag{6.21}$$

and
$$\tilde{\mathbf{L}}_{gS} = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{M}_0 + \varphi_2 \mathbf{M}_0^2 , \tag{6.22}$$

where the φ_i 's (resp. ϕ_i 's) are scalar-valued functions of the three principal invariants of $\mathbf{M}_0$ (resp. $\mathbf{C}_e$). An example of energy function W_c reads:

$$W_c = C_1\left(I_1 I_3^{-1/3} - 3\right) + C_2\left(I_2 I_3^{-2/3} - 3\right) + \frac{1}{2} C_3\left(I_3^{1/2} - 1\right)^2 \tag{6.23}$$

But in the case of *anisotropic* growth (cf. Imatani and Maugin, 2002) we can take

$$\tilde{\mathbf{L}}_{gS} = \mathbf{G}_W(\mathbf{M}_0, \mathbf{W}_0) , \ \mathbf{W}_0 \mathbf{a} = \mathbf{a} \times \mathbf{w}_0, \ \forall \mathbf{a} \tag{6.24}$$

with, e.g.,

$$\mathbf{G}_W(\mathbf{M}_0, \mathbf{N}_0) = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{M}_0 + \varphi_3 \mathbf{N}_0 + \frac{1}{2}\varphi_7\left(\mathbf{M}_0\mathbf{N}_0 + \mathbf{N}_0\mathbf{M}_0\right) , \tag{6.25}$$

where $\mathbf{N}_0$ is such that $\mathbf{W}_0^2 = \mathbf{N}_0 - \mathbf{1}$.

7. MIXTURE THEORY

Such a theory will naturally intervene in the biophysical context with a possible exchange of mass between constituents. The latter, numbered $\alpha = 1,2,..,M$, are supposed to be equipresent in the continuum. In the Eulerian representation we have the following local balance of mass:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha . \nabla\right)\rho_\alpha + \rho_\alpha\left(\nabla . \mathbf{v}_\alpha\right) = m_\alpha \neq 0, \ \alpha = 1,2,..,M \ , \tag{7.1}$$

while with summation over α we obtain

$$\sum_{\alpha=1}^{n} m_\alpha = 0 , \tag{7.2}$$

so that the whole mixture has a classical balance of continuity.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0 \tag{7.3}$$

with

$$\rho = \sum_{\alpha=1}^{M} \rho_\alpha \ , \ \mathbf{v} = \frac{1}{\rho}\sum_{\alpha=1}^{M} \rho_\alpha \mathbf{v}_\alpha \ , \ \mathbf{u}_\alpha := \mathbf{v}_\alpha - \mathbf{v} \ , \ \alpha = 1,2,..,M \ . \tag{7.4}$$

Then (7.1) also reads

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla.(\rho_\alpha \mathbf{v}) = m_\alpha - \nabla.(\rho_\alpha \mathbf{u}_\alpha) \ , \ \alpha = 1,2..,M \ . \tag{7.5}$$

Example: *Fick's diffusion law*. It reads

$$\rho_\alpha \mathbf{u}_\alpha = -K_\alpha \nabla \rho_\alpha . \tag{7.6}$$

A priori the total considered system is not thermodynamically open. But if this were not the case, then, (7.3) would admit a nonzero right-hand side (cf. Equation (6.3) with while the local internal energy balance would read

$$\rho \frac{de}{dt} = t^{ij} d_{ij} - \nabla . \mathbf{q} + \bar{u} \ , \tag{7.7}$$

where $\bar{u}$ would be the internal energy due to the entering mass, so that with a similar source $\bar{h}$ for entropy, the Clausius-Duhem inequality would become

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + t^{ij} d_{ji} - \theta^{-1} \mathbf{q} . \nabla\theta + h \geq 0, \quad h = \bar{u} - \bar{h} \ . \tag{7.8}$$

As an example, we have the evolution of the volume fraction ζ_0 of bone (in a reference configuration free of stresses) due to the entering mass (cf. Epstein, 2012)

$$\frac{d\zeta_0}{dt} = \frac{J\pi}{\gamma_0} \tag{7.9}$$

where γ_0 is the intrinsic density of the undeformed matrix.

A rational introduction to thermodynamically open systems applicable to biomechanics was given by Kuhl and Steinmann (2003).

The true aficionados will appreciate the complexity of a modelling that accounts simultaneously for the possible occurrence of singular surfaces and points (in the development of both volume and surface – e.g., tangential – growth, via the introduction of the second gradient of deformation and the allied geometric non-Riemannian context), the inclusion of exchange of mass between species (theory of mixture) and its application to realistic bio-shapes (horns, seashells) in Ciarletta *et al* (2012, 2013).

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