DOUBLE INCREMENTAL VARIATIONAL PROCEDURE FOR ELASTOPLASTIC COMPOSITES WITH ISOTROPIC AND LINEAR KINEMATIC HARDENING MATRIX REINFORCED BY ELASTIC SPHERICAL PARTICLES

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Keywords : Variational incremental procedure, composites, isotropic and linear kinematic hardening

Abstract

We investigate the nonlinear behavior of elasto-plastic composites with isotropic and linear kinematic hardening. We first rely on the incremental variational principles introduced by Lahellec and Suquet [5]. We also take advantage of an alternative formulation, recently proposed by Agoras et al [1] for visco-plastic composites without hardening, which consists in a double application of the variational procedure of Ponte-Castañeda. We extend in this paper this approach to elasto-plastic composites with combined linear kinematic and isotropic work-hardening. The first application of the variational procedure linearizes the local behavior, including hardening, and leads to a thermo-elastic Linear Comparison Composite (LCC) with a heterogeneous polarization field inside the phases. The second one deals with the heterogeneity of the polarization and results in a new thermo-elastic LCC with a per-phase homogeneous polarization field, which effective behavior can then be estimated by classical linear homogenization schemes. We develop and implement this new incremental variational procedure for composites comprised of linear elastic spherical particles isotropically distributed in an elasto-plastic matrix. The predictions of the model are compared with results available in the literature for cyclic proportional and non-proportional loadings. New results for elasto-plastic composites with combined isotropic and kinematic hardening are also provided. They are in good agreement with the numerical computations we carried out, at both local and macroscopic scales.

1 Introduction

For the last 25 years, a huge interest for predicting the effective response of nonlinear composites has been noting. Most of the work realized on this subject was done with a single potential governing the nonlinear behavior : the free-energy density in the case of hyperelastic materials and the dissipation potential in the case of viscous or rigid-materials. These theories are based on variation principles with linearization schemes and linear homogeneizaton schemes. At the begining, only the first moment was used in these schemes. Thanks to variational arguments, different autors highlighted the importance to work with the second moment per phase which enhance the prediction of the models (Ponte Castañeda [7]). Moreover, Ponte Castañeda [9] has shown that consider both first and second moments per phase allow to have better results thant considering only one of them.

A new stage has been crossed by Lahellec and Suquet [5] who developped a new incremental variational procedure to both local and global scale. The local variational principle lies on the introduction of a unique potential, the condensed incremental potential. This potential is built as the sum of the free-energy density and the dissipation potential which describe the local behavior of GSM (Halphen and Nguyen [2]). From the condensed incremental potential, Lahellec and Suquet determined an effectif incremental variational principle describing the behavior of elastoplastic heterogeneous materials. Thanks to the nonlinear homogenization theory, it is possible to extand this principle to composites comprised of phases governed by a condensed incremental potential. Lahellec and Suquet [5] applied this method to extand the variational procedure introduced by Ponte Castañeda [7] to nonlinear composites comprised of elasto-viscoplastic phases.

In 2013, Lahellec and Suquet [6] proposed a new incremental variational principle to get new estimations of the local and global behavior for composites comprised of elasto-viscoplastic phases with both isotropic and linear kinematic hardening. This new procedure relies on two steps. Firstly, using the variational procedure,

they obtained a secant approximation of the behavior (by linerizing it) leading to a thermoelastic LCC with heterogeneous polarization. Then, this LCC with heterogeneous polarization is homogenized thanks to the new method proposed by Lahellec et al. [4].
More recently, Agoras and al. [1] proposed an alternative formulation of the incremental variation procedure of Lahellec and Suquet [6] to approach the global and local behavior of elasto-viscoplastic composites. This procedure is also composed of two steps. The first one consists in use the linear comparison method developped by Ponte Castañeda [7] to linerized the behavior to get a linerize LCC with non-uniform phase properties. The second one is inpired of the Lahellec et al. [4] method to approximate the linerize LCC with non-uniforme phase properties to a homegenous LCC with uniform properties per phase which permit to obtain the first and second moment of the fields in the phases.
The approach presented in this study is based on the key idea presented by Agoras et al. [1] to handle sequentially the linearization of the local behavior and the accounting for the heterogeneity of the LCC. We propose to extend this idea, initially applied to elasto-viscoplastic composites without hardening, to elastoplastic composites with isotropic and linear kinematic hardening. Firstly we described the local behavior and the local and global variational principles by inspiring of Lahellec and Suquet [5] [6]. Then, we developped the incremental variation procedure based on the Agoras et al. [1] work. For that, we obtained first the linerize LCC with heterogeneous coefficients per phase, then, we approximated this LCC to a homogeneous one. This method is applies to the case of elastoplastic composites reinforced by spherical linear elastic particles distributed isotropically. Finally, we compared our model to the RVP one Lahellec and Suquet [6] for ideally-plastic matrix and matrix with kinematic hardening under a axial loading. Moreover, we proposed new data. Indeed, we computed Finite Element Method simulations on an elastoplastic composite with both isotropic and linear kinematic hardening.

2 Local behavior and incremental variational principles

We consider a Representative Volume Element (RVE) of a N-phases composite material Ω with $\Omega^{(r)}$ the occupied volume by the phase r $(r = 1, \ldots, N)$. The phases are GSM having an elasto-plastic local behavior with linear kinematic hardening and non linear isotropic hardening which is conventionally described by the theory J_2 of the plasticity. This corresponds to a material with internal variables $\boldsymbol{\alpha} = (\boldsymbol{\varepsilon}^p, p)$ describing irreversible phenomena, where $\boldsymbol{\varepsilon}^p$ the plastic strain field and p the accumulate plastic strain field, with two convex potentials.
The first one is the free-energy density $w^r(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, p)$ where $\boldsymbol{\varepsilon}$ denotes the local strain field

$$w^{(r)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbb{L}^{(r)} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}\,\boldsymbol{\varepsilon}^p : \mathbb{H}^{(r)} : \boldsymbol{\varepsilon}^p + \hat{w}^{(r)}(p), \quad \text{with } \frac{dw^{(r)}}{dp}(p) = R^{(r)}(p) \quad (1)$$

with $\mathbb{L}^{(r)}$ the elasticity tensor, $\mathbb{H}^{(r)}$ the kinematix hardening second order tensor, $\hat{w}^r(p)$ a scalar function which depend on p representing the accumulated plastic strain due to the isotropic hardening and $R(p)$ is a scalar function characterizing the isotropic hardening.
The second potential is the dissipation potential $\varphi^{(r)}(\dot{\boldsymbol{\alpha}})$ which was obtained by Ladeveze in 1996 [3]

$$\varphi^{(r)}\left(\dot{\boldsymbol{\varepsilon}}^p, \dot{p}\right) = \phi^{(r)}\left(\dot{\varepsilon}^p_{eq}\right) + \Phi_{\mathcal{C}}(\dot{\boldsymbol{\varepsilon}}^p, \dot{p}) \quad (2)$$

with

$$\phi^{(r)}(\dot{\varepsilon}^p_{eq}) = \sigma_y^{(r)}\,\dot{\varepsilon}^p_{eq} \quad \text{and} \quad \Phi_{\mathcal{C}}(\dot{\boldsymbol{\varepsilon}}^p, \dot{p}) = \begin{cases} 0 & \text{if } (\dot{\boldsymbol{\varepsilon}}^p, \dot{p}) \in \mathcal{C} \\ +\infty & \text{otherwise} \end{cases} \quad (3)$$

where $\Phi_{\mathcal{C}}$ denotes the indicator function on the convex set $\mathcal{C} = \{(\dot{\boldsymbol{\varepsilon}}^p, \dot{p}) / g(\dot{\boldsymbol{\varepsilon}}^p, \dot{p}) = \dot{\varepsilon}^p_{eq} - \dot{p} \leq 0\}$.

As explained in Lahellec and Suquet [5], an approximation of the constitutive equations for classic GSM can be obtained by means of the following equations using an implicit Euler-scheme

$$\boldsymbol{\sigma}_{n+1} = \frac{\partial\, w^{(r)}}{\partial\, \boldsymbol{\varepsilon}_{n+1}}(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\alpha}_{n+1}) \qquad \frac{\partial\, w^{(r)}}{\partial\, \boldsymbol{\alpha}_{n+1}}(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\alpha}_{n+1}) + \frac{\partial\, \varphi}{\partial\, \dot{\boldsymbol{\alpha}}_{n+1}}\left(\frac{\boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n}{\Delta t}\right) = 0 \quad (4)$$

The time interval of study $[0, T]$ is discretized into N time interval and we defined $\Delta t = t_{n+1} - t_n$. To simplify the notations, we will omit the index $n + 1$ for the variables computed at the time t_{n+1} (i.e. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{n+1}$).

As in Lahellec and Suquet [5] we introduce the condensed incremental potential J which is defined as the sum of the free-energy $w^{(r)}$ and the dissipation potential $\varphi^{(r)}$. This potential is nonuniform per phase. Moreover, we introduce the condensed free-ernergy w_Δ and the condensed effective free-energy $\tilde{w}_\Delta$ defined as

$$\tilde{w}_\Delta(\boldsymbol{E}) = \inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle=\boldsymbol{E}} \langle w_\Delta(\boldsymbol{\varepsilon})\rangle = \inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle=\boldsymbol{E}} \left\langle \inf_{\boldsymbol{\alpha}=(\boldsymbol{\varepsilon}^p,p)} J(\underline{x},\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,p) \right\rangle \tag{5}$$

Lahellec and Suquet [5] show, thanks to the relation $\boldsymbol{\sigma} = \partial w_\Delta/\partial\boldsymbol{\varepsilon}$ and Hill's lemma, that the macroscopic stress $\boldsymbol{\Sigma} = \langle\boldsymbol{\sigma}\rangle$ is given by

$$\boldsymbol{\Sigma} = \frac{\partial\tilde{w}_\Delta}{\partial\boldsymbol{E}}(\boldsymbol{E}) \tag{6}$$

3 Application of the double incremental variational procedure to composites with hardening

The incremental variational procedure, proposed by Lahellec and Suquet [5], leads to the definition of a linear comparison composite (LCC) with homogeneous polarization per phase, characterized by a free energy w_0. The variational procedure is applied once to deal with both the nonlinearity of the phases and the heterogeneity of the LCC within the phases. In this method, when defining the linearized incremental potential J_0, it is not easy to exhibit the adequate closed form expression of J_0, since two stages, namely the linearization of the behavior and the handling of the heterogeneity of the LCC, are melted. Agoras et al. [1] introduced a more systematic method in which the linearization of the behavior and the heterogeneity of the LCC within the phases are adressed in two separate steps. The first step makes use of the variational procedure of Ponte Castañeda [7] to obtain a LCC with heterogeneous eigenstrains within the phases. The second one which also relies on the variational procedure makes use of the method proposed by Lahellec et al. [4] to reduce the resulting problem to a different LCC with now homogeneous properties.
In the present study, we reformulate the key idea proposed by Agoras et al. [1] - which consists to deal with sequentially the linearization of the local behavior and the heterogeneity of the resulting LCC - initially applied to ideally-plastic phases to the context of elastoplastic composites with hardening phases. Both isotropic and linear kinematic hardening are considered.
To clearly distinguish our approach from previous works of the literature dealing with the same issue, we first emphasize that our formulation relies on the initial variational incremental principle introduced by Lahellec and Suquet [5] in a total form and not on the modified version proposed by Agoras et al. [1] in a rate form, i.e. based on the strain rate $\dot{\varepsilon}$ instead of the total strain ε. We also stress that our approach is established in primal form (based on $w(\boldsymbol{\varepsilon})$ and $\varphi(\dot{\boldsymbol{\alpha}})$) while a dual formulation was considered by Agoras et al. [1].

Firstly, owing to the non quadratic character of the dissipation potential $\varphi^{(r)}$ the incremental potential J is difficult to homogenize. To bypass this difficulty a linearized incremental potential $J_L^{(r)}$ is introduced in order to approach the dissipation potential $\varphi^{(r)}$ by a quadratic function of $\dot{\boldsymbol{\varepsilon}}^p$ which depends on a viscosity $\eta_{\varepsilon^p}^{(r)}$ which is uniform in phase r.

$$J_L^{(r)}\left(\underline{x},\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,p,\eta_{\varepsilon^p}^{(r)}\right) = w^{(r)}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,p) + \frac{\eta_{\varepsilon^p}^{(r)}}{\Delta t}\left(\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p\right):\left(\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p\right) \tag{7}$$

A key idea of the variational procedure is to add and subtract to the potential J the potential J_L ($J = J_L + J - J_L$), such that the first term J_L can be homogenized by classical methods for linear media thanks to the quadratic part, while the difference $J - J_L$ can still be evaluated separately.
We obtain an expression of $\tilde{w}_\Delta$ from (5) which depends on J_L and ΔJ. Moreover, as $\Phi_{\mathcal{C}}(\dot{\boldsymbol{\varepsilon}}^p,\dot{p}) = +\infty$ when $g(\dot{\boldsymbol{\varepsilon}}^p,\dot{p}) > 0$, the infimum in this expression of $\tilde{w}_\Delta$ is obtained under the condition $g\left(\frac{\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p}{\Delta t},\frac{p-p_n}{\Delta t}\right) \leq 0$. Following the work of Lahellec and Suquet [5] we approximate the effective free-energy $\tilde{w}_\Delta$ by $\tilde{w}_\Delta^{var}$ as

$$\tilde{w}_\Delta(\boldsymbol{E}) \approx \tilde{w}_\Delta^{var}(\boldsymbol{E},\{\eta_{\varepsilon^p}\}) = \inf_{\langle\boldsymbol{\varepsilon}\rangle=\boldsymbol{E}} \left\{ \inf_{(\boldsymbol{\varepsilon}^p,p)/h(\boldsymbol{\varepsilon}^p,p)\leq 0} \langle J_L(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,p,\{\eta_{\varepsilon^p}\})\rangle + \left\langle \operatorname*{stat}_{(\dot{\boldsymbol{\varepsilon}}^p,\dot{p})/g(\dot{\boldsymbol{\varepsilon}}^p,\dot{p})\leq 0} \Delta J_{bis}(\dot{\boldsymbol{\varepsilon}}^p,\{\eta_{\varepsilon^p}\}) \right\rangle \right\} \tag{8}$$

where h is a rewrite of the function g which depends on $\boldsymbol{\varepsilon}^p$ and p, the potential $\Delta J_{bis} = \Delta J - \Delta t \Phi_{\mathcal{C}}$ and the notation $\{\eta_{\varepsilon^p}\}$ stands for the set $\{\eta_{\varepsilon^p}^{(1)}, ..., \eta_{\varepsilon^p}^{(N)}\}$.

Now, we have to compute the stationarity conditions :

— The stationarity of ΔJ_{bis} with respect of $(\dot{\varepsilon}^p, \dot{p})$ leads to :

$$\eta_{\varepsilon^p}^{(r)} = \frac{1}{3}\frac{\partial \phi^{(r)}}{\partial \dot{\varepsilon}_{eq}^p}\left(\dot{\varepsilon}_{eq}^p\right) = \frac{\sigma_y^{(r)}}{3\,\dot{\varepsilon}_{eq}^p} = \eta_{\phi,sct}^{(r)}\left(\dot{\varepsilon}_{eq}^p\right) \tag{9}$$

— The stationarity of $\tilde{w}_\Delta^{var}$ with respect of $\eta_{\varepsilon^p}^{(r)}$ gives :

$$\dot{\varepsilon}_{eq}^p = \overline{\overline{\dot{\varepsilon}^p}}^{(r)} = \sqrt{\frac{2}{3}\left\langle \left(\frac{\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_n^p}{\Delta t}\right) : \left(\frac{\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_n^p}{\Delta t}\right)\right\rangle^{(r)}} \Rightarrow \eta_{\phi,sct}^{(r)}\left(\dot{\varepsilon}_{eq}^p\right) = \eta_{\phi,sct}^{(r)}\left(\overline{\overline{\dot{\varepsilon}^p}}^{(r)}\right) \tag{10}$$

— The stationarity of J_L with respect of $(\boldsymbol{\varepsilon}^p, p)$ delivers :

1. The cumulated plastic strain p is uniform per phase and defines as : $p^{(r)} = p_n^{(r)} + \Delta t\, \overline{\overline{\dot{\varepsilon}^p}}^{(r)}$
2. The expression of the viscosity : $\eta_{\varepsilon^p,sct}^{(r)}\left(\overline{\overline{\dot{\varepsilon}^p}}^{(r)}\right) = \frac{\sigma_y^{(r)} + R^{(r)}\left(p^{(r)}\right)}{3\,\overline{\overline{\dot{\varepsilon}^p}}^{(r)}}$
3. The expression of the plastic strain field : $\boldsymbol{\varepsilon}^p = \left[\mathbb{K} : \mathbb{L}^{(r)} + \mathbb{H}^{(r)} + 2\,\frac{\eta_{\varepsilon^p,sct}^{(r)}}{\Delta t}\mathbb{K}\right]^{-1} : \left[\mathbb{K} : \mathbb{L}^{(r)} : \boldsymbol{\varepsilon} + 2\,\frac{\eta_{\varepsilon^p,sct}^{(r)}}{\Delta t}\boldsymbol{\varepsilon}_n^p\right]$

From now, by introducing the expression of $\boldsymbol{\varepsilon}^p$ in the definition of $J_L^{(r)}$ cf. Eq.(7), it is shown that the energy $w_L^{(r)}$ takes the form

$$w_L^{(r)}(\underline{x}, \boldsymbol{\varepsilon}) = \inf_{(\boldsymbol{\varepsilon}^p, p)} J_L^{(r)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, p) = J_L^{(r)}\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_n^p\right)\right) = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{L}_L^{(r)} : \boldsymbol{\varepsilon} + \boldsymbol{\tau}_L^{(r)}(\underline{x}) : \boldsymbol{\varepsilon} + \frac{1}{2} f_L^{(r)}(\underline{x}) \tag{11}$$

Such energy corresponds to a LCC with heterogeneous intraphase polarization, $\boldsymbol{\tau}_L^{(r)}$ and $f_L^{(r)}$ depending on $\underline{x}$ through $\boldsymbol{\varepsilon}_n^p$. The effective energy of this LCC is defined as $\tilde{w}_L(\boldsymbol{E}) = \inf_{\langle\boldsymbol{\varepsilon}\rangle = \boldsymbol{E}} \langle w_L(\boldsymbol{E})\rangle$. Due to the stationarity conditions of ΔJ_{bis} with respect of $\dot{\varepsilon}_{eq}^p$ and of $\tilde{w}_\Delta^{var}$ with respect of $\eta_{\varepsilon^p}^{(r)}$, one gets

$$\boldsymbol{\Sigma} = \frac{\partial \tilde{w}_\Delta}{\partial \boldsymbol{E}}(\boldsymbol{E}) = \frac{\partial \tilde{w}_L}{\partial \boldsymbol{E}}(\boldsymbol{E}) \tag{12}$$

Secondly, we implement the above procedure proposed by Lahellec et al. [4] for an elastoplastic behavior with isotropic and linear kinematic hardening, for which $w_L^{(r)}(\underline{x}, \boldsymbol{\varepsilon})$ is defined by (11) with constant moduli $\mathbb{L}_L^{(r)}$ and heterogeneous polarization $\boldsymbol{\tau}(\underline{x})$ and $f_L^{(r)}(\underline{x}) \neq 0$. Thus, the variational procedure will be applied to get a classic LCC with an energy $w_0^{(r)}$ (approaching the LCC with the energy $w_L^{(r)}$) defined as

$$w_0(\underline{x}, \boldsymbol{\varepsilon}) = \sum_{r=1}^{N} w_0^{(r)}(\boldsymbol{\varepsilon})\,\chi^{(r)}(\underline{x}), \quad \text{with } w_0^{(r)}(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{L}_0^{(r)} : \boldsymbol{\varepsilon} + \boldsymbol{\tau}_0^{(r)} : \boldsymbol{\varepsilon} + \frac{1}{2} f_0^{(r)}. \tag{13}$$

The effective free energy of this LCC is given by Willis [10] as

$$\tilde{w}_0(\boldsymbol{E}) = \inf_{\langle\boldsymbol{\varepsilon}\rangle = \boldsymbol{E}} \langle w_0(\underline{x}, \boldsymbol{\varepsilon})\rangle = \frac{1}{2}\boldsymbol{E} : \tilde{\mathbb{L}}_0 : \boldsymbol{E} + \tilde{\boldsymbol{\tau}}_0 : \boldsymbol{E} + \frac{1}{2}\tilde{f}_0. \tag{14}$$

From the procedure proposed by Lahellec et al., we obtain, thanks to the relations of Ponte Castañeda and Suquet [8] the first and second-order moments of $\boldsymbol{\varepsilon}_0$. Moreover, Ponte Castañeda and Suquet [8] demonstrated that the first and second-order moment of the strain field inside the LCC $w_L^{(r)}$ can be estimated by those of the LCC $w_0^{(r)}$. So, one gets

$$\overline{\boldsymbol{\varepsilon}_L}^{(r)} = \overline{\boldsymbol{\varepsilon}_0}^{(r)}, \quad \langle \boldsymbol{\varepsilon}_L \otimes \boldsymbol{\varepsilon}_L\rangle^{(r)} = \langle \boldsymbol{\varepsilon}_0 \otimes \boldsymbol{\varepsilon}_0\rangle^{(r)} \tag{15}$$

The effective behavior of the nonlinear composite can be now computed from the estimate

$$\boldsymbol{\Sigma} = \frac{\partial \tilde{w}_{\Delta}}{\partial \boldsymbol{E}}(\boldsymbol{E}) = \frac{\partial \tilde{w}_{L}}{\partial \boldsymbol{E}}(\boldsymbol{E}) = \frac{\partial \tilde{w}_{0}}{\partial \boldsymbol{E}}(\boldsymbol{E}) = \tilde{\mathbb{L}}_0 : \boldsymbol{E} + \tilde{\boldsymbol{\tau}}_0 \tag{16}$$

Finally, we applied this procedure in the case of an elasto-plastic composites reinforced by spherical linear elastic particles distributed isitropically (N=2). Making use of the Hashin Strikman estimates, it is possible to compute the effective behavior of the composite as well as the first and second-order moment of the different fields in each phase.

4 Applications and discussions

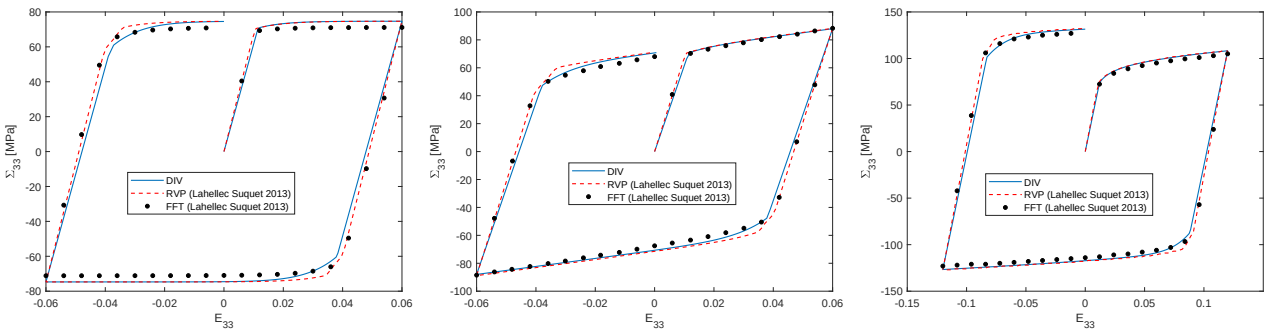
4.1 Case of ideally-plastic matrix and matrix with isotropic or linear kinematic hardening

This section deals with composite materials composed of an elsatoplastic matrix (ideally-plastic and with isotropic or linear kinematic hardening) reinforced by elastic spherical inclusions which has linear ans isotropic behavior. These particles are randomly and isotropically distributed in the matrix.
The composites studied are summited to the same loading as Lahellec and Suquet [6]. These authors have considered a macroscopic strain tensor $\boldsymbol{E}$ in the form of an isochoric extension $E_{33}(t)$ in the axial direction

$$\boldsymbol{E}(t) = E_{33}(t)\left(-\frac{1}{2}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\right) \tag{17}$$

Results of our model will be compared to the prediction of the RVP one developped by Lahellec and Suquet [6] and to Fast Fourier Transform (FFT) simulations carried by [6] on a REV comprised of 50 spherical inclusions randomly distributed in the matrix.
The same materials parameters as in Lahellec and Suquet [6] are considered. The values of the phases parameters are the following

$$\begin{aligned} &\text{Inclusion}: \ c^{(1)} = 0.17, \ \ E^{(1)} = 16.368\,\text{GPa}, \ \ \nu^{(1)} = 0.4999999 \\ &\text{Matrix}: \ E^{(2)} = 8.1846\,\text{GPa}, \ \ \nu^{(2)} = 0.4999999, \ \ \sigma_0^{(2)} = 100\,\text{MPa} \end{aligned} \tag{18}$$



(a) (b) (c)

FIGURE 1: Macroscopic axial stress under a radial loading of the composite comprised of elastic spherical particles. Comparison between the actual model (solid line), the RVP model [6] (dotted line) and full-field simulations [6] (dots). (a) Ideally-plastic matrix, (b) Matrix with kinematic hardening, (c) Matrix with isotropic hardening.

The figure 1 representes the variation of the macroscopic stress during a loading for ideally-plastic matrix $\left(\mathbb{H}^{(2)} = a^{(2)}\mathbb{K} = 0 \text{ and } R^{(2)}(p) = 0\right)$, matrix with linear kinematic hardening $\left(a^{(2)} = 300\,\text{MPa} \text{ and } R^{(2)}(p) = 0\right)$ and matrix with isotropic hardening $\left(a^{(2)} = 0\,\text{MPa} \text{ and } R^{(2)}(p) = \beta^{(2)}p^{\gamma^{(2)}} \text{ where } \beta^{(2)} = 100\,\text{MPa} \text{ and } \gamma^{(2)} = 0.4\right)$. We note, in the three cases, a good agreement between our model and the RVP one for the macroscopic axial stress E_{33}. As the RVP model ours is able to reproduce the Bauschinger effect. It is noticed that the elastic

state slope between our model and the RVP one is different. This comes from the material coefficients. Indeed, Lahellec and Suquet considered compressible composite and not incompressible one, so our shear moduli are differents from them. Moreover, the model decribed the trend of the FFT simulations. Due to the approximation of the dissipative potential $\sigma_y^{(r)}\dot{\varepsilon}_{eq}^p$ by a quadratic potential $\frac{\eta_{\varepsilon^p}^{(r)}}{\Delta t}(\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_n^p) : (\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_n^p)$ the model overestimates the macroscopic stress.

4.2 Multiple cycles loadings for a matrix with isotropic hardening

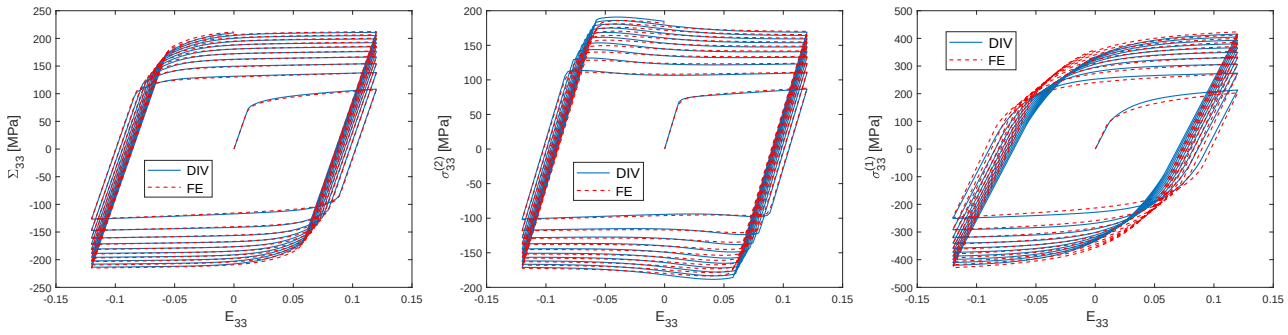
In this Section, we explore the predictions of the DIV model when several loading cycles are considered. From now on, we will only study an elastoplastic matrix with an isotropic hardening characterized by the power law

$$R^{(2)}(p) = \beta^{(2)} p^{\gamma^{(2)}}. \tag{19}$$

We also consider the reinforced composite dealt with by Lahellec and Suquet [6] whose material parameters are defined by Eq. (18) for the elastic properties of the phases and by the following parameters for the isotropic hardening

$$\beta^{(2)} = 100\,\text{MPa}, \quad \gamma^{(2)} = 0.4\,. \tag{20}$$

For an elastic ideally-plastic matrix or an elastoplastic matrix with linear kinematic hardening, it is found that the macroscopic and local responses are stabilized as soon as the first cyclic loading occurs. This is not the case for an elastoplastic matrix with isotropic hardening as it can be seen on Fig. 2 which reports for 10 cycles the evolution of the axial stress and the averages of the axial stress over the phase for both the DIV approach and FE periodic simulations. As in former sections, the FE simulations are carried out on a periodic cubic cell made of a single spherical elastic inclusion embedded in an elastic ideally-plastic matrix. On a whole, a close agreement is observed between the DIV approach and the FE simulations, especially for the macroscopic axial stress and for the average axial stress in the matrix. As observed on the macroscopic axial stress, the asymmetry characterising the Bauschinger effect increases continuously with the number of cycles, going from 24,64 Mpa for the first cycle to 40,6 Mpa for the tenth cycle. This evolution of the asymmetry characterising the Baushinger effect is accurately captured by the DIV approach. Lastly, it can be seen that the macroscopic and local responses tends to a limit cycle, thus showing that the DIV approach, in agreement with the FE simulations, predicts a plastic shakedown when the plastic matrix exhibits isotropic hardening.



(a) (b) (c)

FIGURE 2: Elastically reinforced composite submitted to 10 cycle radial loadings. Case of an elasto-plastic matrix with isotropic work-hardening. $c^{(1)} = 0.17$. (a) Macroscopic axial stress, (b) Average axial stress in the matrix, (c) Average axial stress in the inclusion.

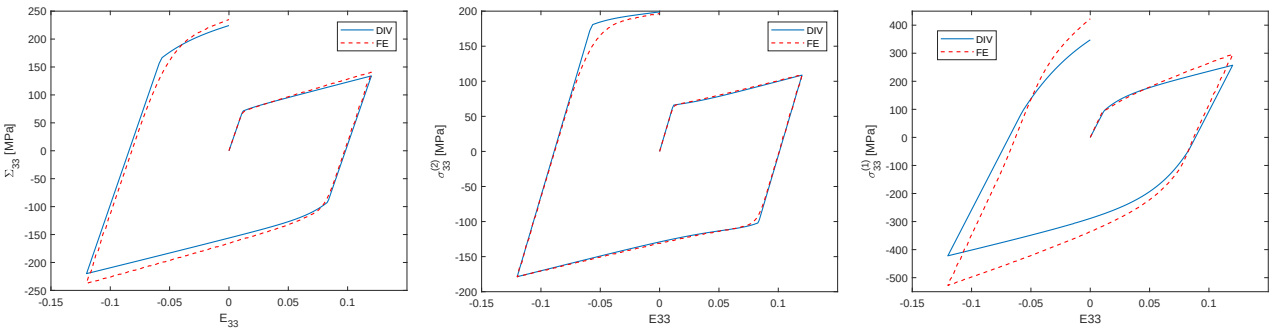
4.3 Case matrix with both isotropic and linear kinematic hardening

This section proposes new data for a composite comprised of an elastoplastic matrix with both kinematic and isotropic hardening reinforced by elastic spherical particles. The model is compared to Finit Elements Method (FEM) simulations. The function characterizing the isotropic hardening $R^{(2)}$ is defined as :

$$R^{(2)}(p) = \left(RM - \sigma_y^{(2)}\right)\left(1 - e^{-Bp}\right) \tag{21}$$

where B and RM two parameters which define the isotropic hardening. During the MEF computations, an eighth of a hree dimentional cubic cell with a spherical inclusion in its center was considered. Conditions of symmetry and periodicity have been set on this cell. During the computations the phases parameters were chossen as the ones already defined in (18), and the parameters for the isotropic and kinematic hardening were fixed as

$$RM = 2.1\,\text{GPa}, \quad B = 0.26, \quad a^{(2)} = 100\,\text{MPa} \tag{22}$$



(a) (b) (c)

FIGURE 3: Composite comprised of matrix with both isotropic and linear kinematic hardening and elastic spherical particles under a radial loading. Comparison between the actual model (solid line) and FEM simulations (dotted line). (a) Macroscopic stress, (b) Average stress in the matrix, (c) Average stress in the inclusion.

The composite was also submited to an axial loading. We observe in the Fig.3 that the trend of the average stress over the matrix is exactly the same as the FEM, excepted for the Bauschinger effect which is overestimated by the model, nevertheless, it is able to reproduce it. On the other hand, it is noted for the average over the inclusion that the trend is the same as the FEM but the model underestimate the FEM with a maximum error of about 11%. Moreover the slope between the model and the FEM is different. It is assumed that this difference comes from the anisotropy in the inclusion which is due to the geometry considered and macroscopic loading applied. These observations explain the reason why the macroscopic stress has the same trand as the MEF but it underestimate it with a maximum error of about 3%. However, it is able to capture the Bauschinger effect.

5 Conclusion

In this study we proposed an incremental variational procedure to approach the effective behavior of an elastoplastic composite with both isotropic and linear kinematic hardening. This method extands the variational procedure proposed by Agoras et al. [1]. We considered composites comprised of GSM phases which are defined with two convex potentials (free-energy and dissipation potential). Following Lahellec and Suquet [5] and [6], the local and macroscopic variational principles are obtained with the introduction of an incremental potential. Referring to Ponte Castañeda [7] and Agoras et al. [1] this potential is aproached by a linearized one which is possible to homogenize thanks to its quadratic part. Into taking advantage of the linear comparaison method of Lahellec et al. [4], the LCC $\tilde{w}_L$ can be homogenized to get a homogeneous LCC. From this method it is possible to compute the expressions of the first and second moment of ε_0.
When possible, the model predictions have been compared to the RVP ones (Lahellec and Suquet [6]) and to FFT simulations [6] under an axial loading. We noted that for ideally-plastic, kinematic hardening and isotropic hardening a good agreement between the model, the RVP one and the FFT simulations for the macroscopic strain. It is noted that the effective stress of the two model overestimates the numerical simulations. Finally, new data on composites either with combined isotropic and kinematic hardening or submitted to several loading cycles have been proposed. Again, the model prediction were compared to FEM simulations. We noted that the model reproduces the FEM data for the macroscopic stress and the average stress in the matrix. A different slope was observed for the average stress in the inclusion ; we believe that this difference comes from the anisotropy in the inclusion due to the geometry and the and to the stress apply by the matrix on the inclusion during the MEF simulations.

Références

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