A BIOLOGICAL MODEL DESCRIBING VAPOUR AND DROPLETS

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Résumé

Delay equations (e.g., Delay Differential Equations) are fundamental in many scientific areas such as biology, medicine, engineering and physics as the introduction of the delay can provide a more realistic mathematical model incorporating the information from the past in the evolution law. Due to the complexity of the analytical investigation of the properties of delay equations, it is necessary to use suitable numerical methods such as the pseudospectral discretization approach which allows to transform the initial model into a system of Ordinary Differential Equations, whose dynamics can be studied with some available softwares for ODEs. Whereas in the case of finite delay we have convergence results [1], in the case of infinite delay numerical evidences are achieved [4], but the proof of convergence is still ongoing.

This work is dedicated to a new biological model introduced by the professor Mats Gyllenberg of the University of Helsinki and currently studied by his PhD student Eugenia Franco. This model describes the evolution of water droplets in a vapour and it can be mathematically represented by a DDE with infinite delay. The latter has some features in common with the Daphnia model introduced in the paper [2].

We consider a finite mass of vapour which contains a finite number of water droplets of different sizes. There are only three phenomena : birth, growth and death of the droplets. Moreover, we assume that the growth of a droplet is determined by a growth function and that there is a unique size-at-birth. The model is described by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \frac{\alpha}{x_0}\int_0^{+\infty}\left[\mu\big(X(a;V_t)\big)\,X(a;V_t) - g\big(X(a;V_t),V(t)\big)\right]\mathcal{F}(a;V_t)\,V_t(-a)\,\mathrm{d}a - \alpha V(t) \tag{1}$$

where $V(t)$ is the mass of the vapour per unit of space at time t, V_t is the history of the vapour, α is the condensation rate, x_0 is the size-at-birth of a droplet, $X(a;V_t)$ is the mass of a droplet at age a, $g(X,V)$ is the growth rate, $\mu(X)$ is the death rate and $\mathcal{F}(a;V_t)$ is the survival probability until age a.

It is possible to prove that the mass conservation holds, i.e., the sum of the concentration of the vapour and of the droplets (that is the total mass concentration) is constant. Moreover, all the constants are equilibria of the DDE (1), i.e., we have infinite equilibria.

In order to fix a finite number of equilibria and to study how an equilibrium changes when a parameter varies, we propose to add a term corresponding to the mass conservation law with a fixed value of the total mass concentration E into the equation (1) :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \frac{\alpha}{x_0}\int_0^{+\infty}&\left[\mu\big(X(a;V_t)\big)\,X(a;V_t) - g\big(X(a;V_t),V(t)\big)\right]\mathcal{F}(a;V_t)\cdot\\ &\cdot V_t(-a)\mathrm{d}a - \alpha V(t) + E - V(t) - \frac{\alpha}{x_0}\int_0^{+\infty} X(a;V_t)\,\mathcal{F}(a;V_t)\,V_t(-a)\,\mathrm{d}a\,,\end{aligned} \tag{2}$$

In the Poster some continuation plots will be shown. They are obtained applying the pseudospectral approach and using the software MATCONT. In the examples we choose a constant death rate, a particular function for the growth rate and we fix the values of the parameters. In all the simulations the dynamics will be very simple : there is one equilibrium which remains asymptotically stable when a parameter varies, i.e., there are no bifurcation points. We could probably get more complex dynamics considering other functions for the growth rate $g(x,V)$, considering a not constant death rate μ or adding other phenomena in the formulation of the model.

Références

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