



# Sur Quelques Travaux en Homogénéisation avec Inertie (et réciproquement)

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jointly with Rémi Cornaggia<sup>2</sup>, Bruno Lombard<sup>1</sup>

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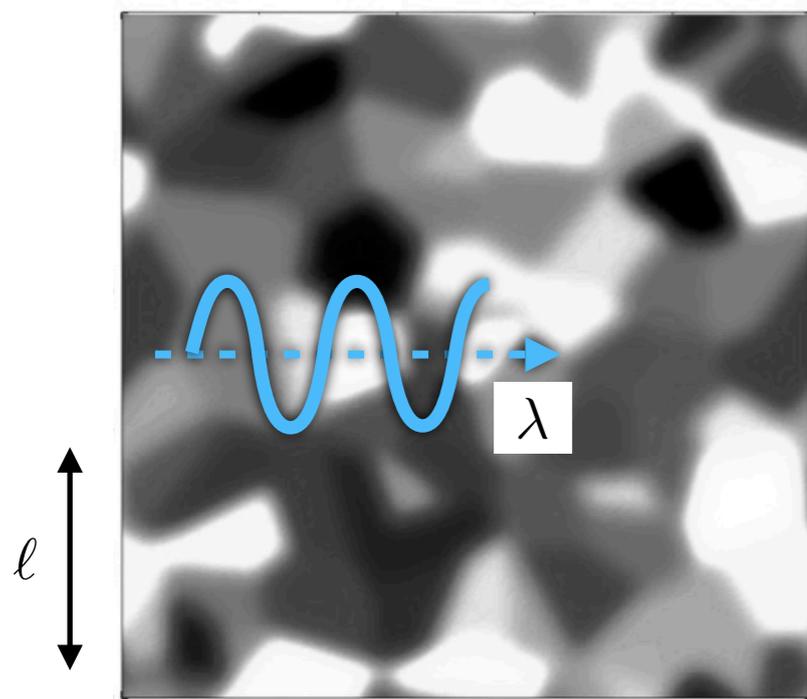
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# Waves in microstructured media

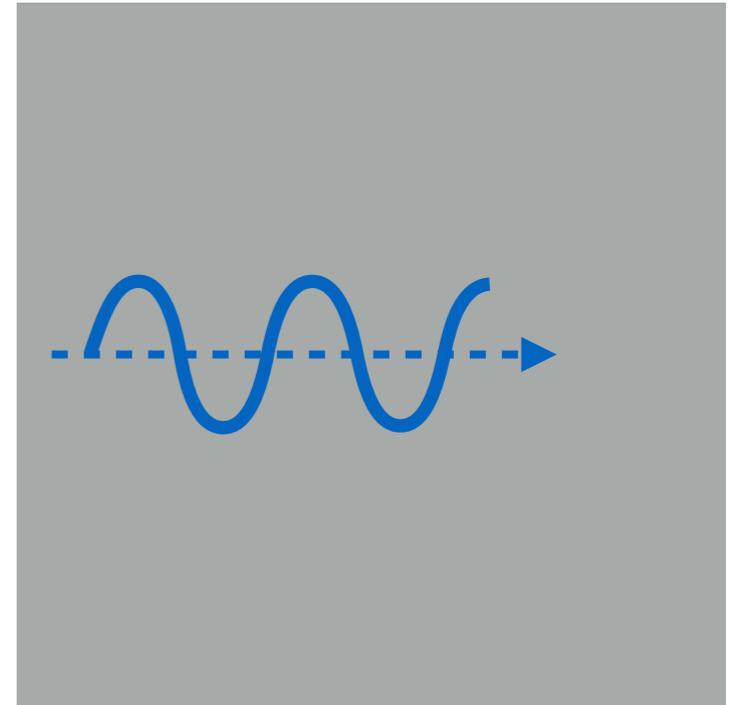


$\rho(\mathbf{x}), \mathbf{L}(\mathbf{x})$

mass density, elastic moduli



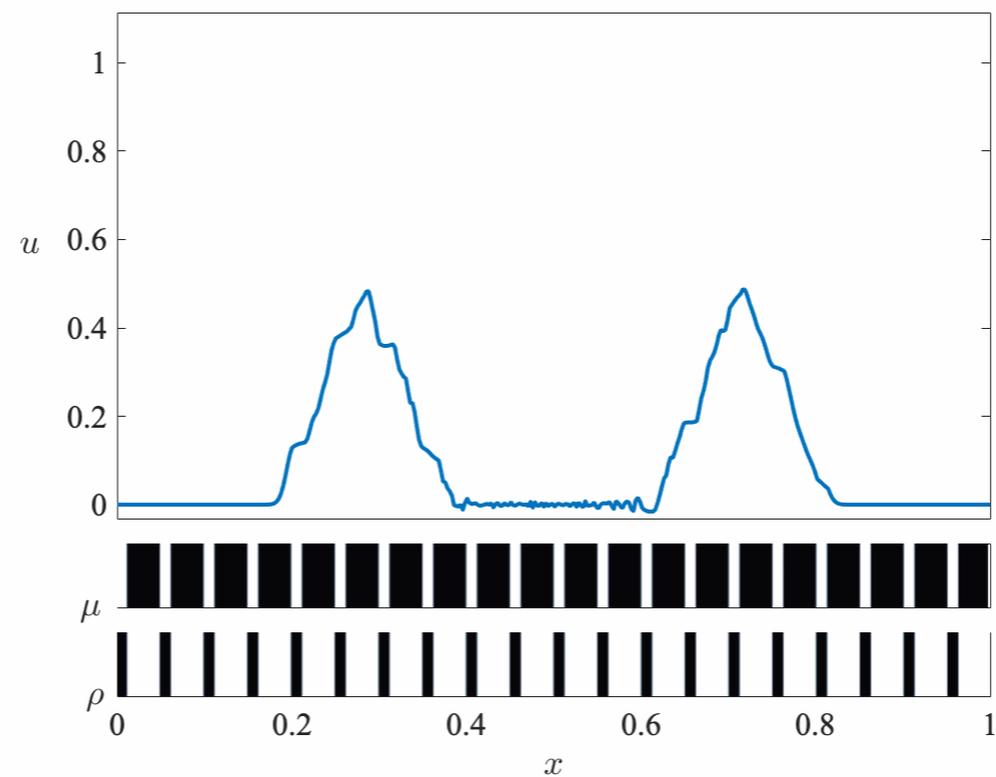
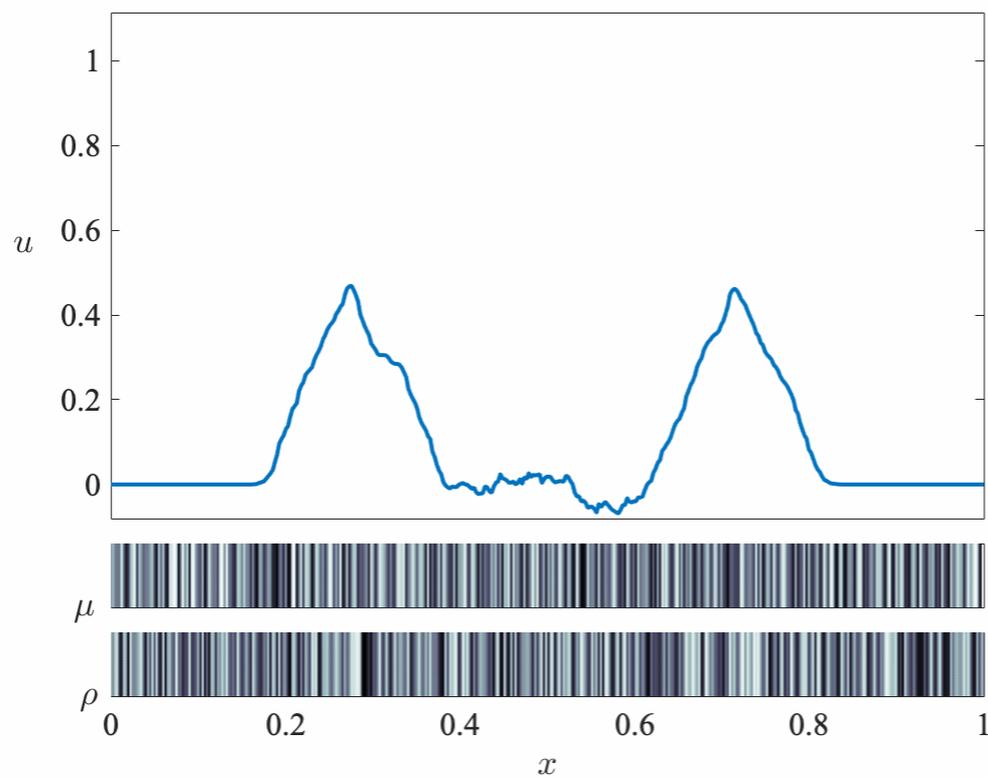
Homogenization



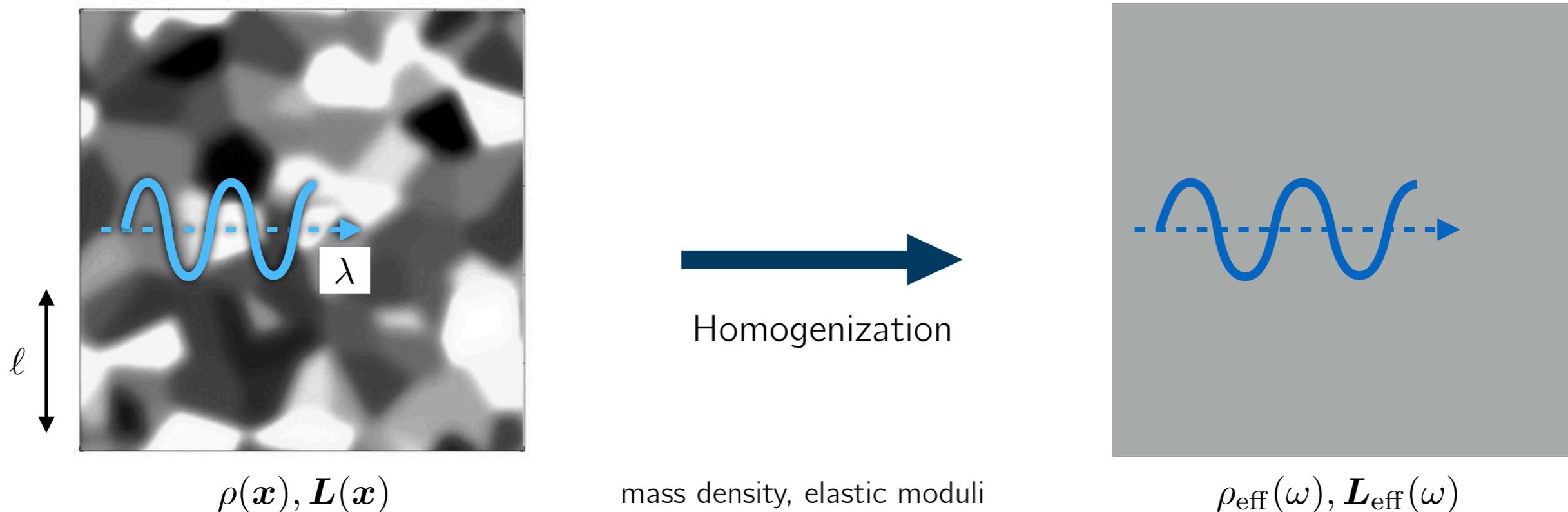
$\rho_{\text{eff}}(\omega), \mathbf{L}_{\text{eff}}(\omega)$

- ▶ **Examples:** 1D, scalar wave (random or periodic)

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial}{\partial x} \left( \mu(x) \frac{\partial u}{\partial x}(x, t) \right)$$



# Waves in microstructured media



## Questions:

- ▶ Knowing  $\rho(\mathbf{x}), \mathbf{L}(\mathbf{x})$  : computation (analytically or numerically) of  $\rho_{\text{eff}}, \mathbf{L}_{\text{eff}}$ ?
  - periodic homogenization, FFT-based computations
  - higher-order homogenization (asymptotic expansions)
- ▶ Computation of dispersion relation characterizing the microstructure
- ▶ Given effective constitutive relations  $\rho_{\text{eff}}(\omega), \mathbf{L}_{\text{eff}}(\omega)$ : simulation of wave propagations?
- ▶ For objective  $\rho_{\text{eff}}, \mathbf{L}_{\text{eff}}$ : computation of optimal  $\rho(\mathbf{x}), \mathbf{L}(\mathbf{x})$ ?

# Macroscopic behavior

- ▶ Leading-order approximation:

$$u = u_0 + O(\eta) \quad \text{with} \quad \eta = \frac{\ell}{\lambda} \ll 1$$

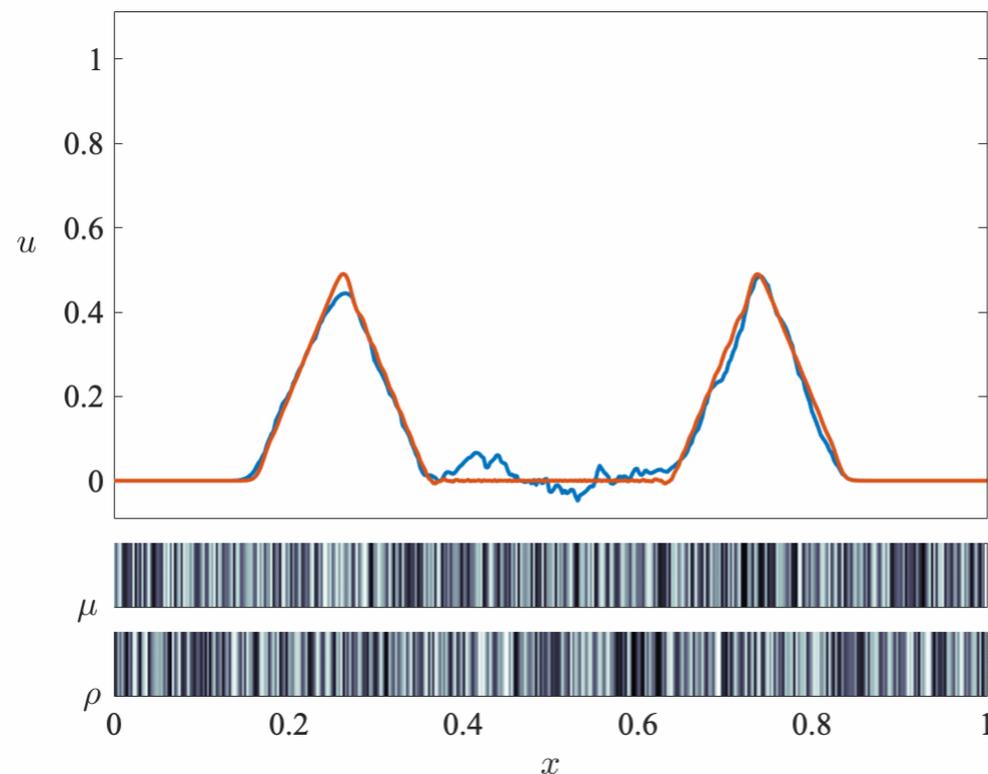
Starting equation (full-field):

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial}{\partial x} \left( \mu(x) \frac{\partial u}{\partial x}(x, t) \right)$$

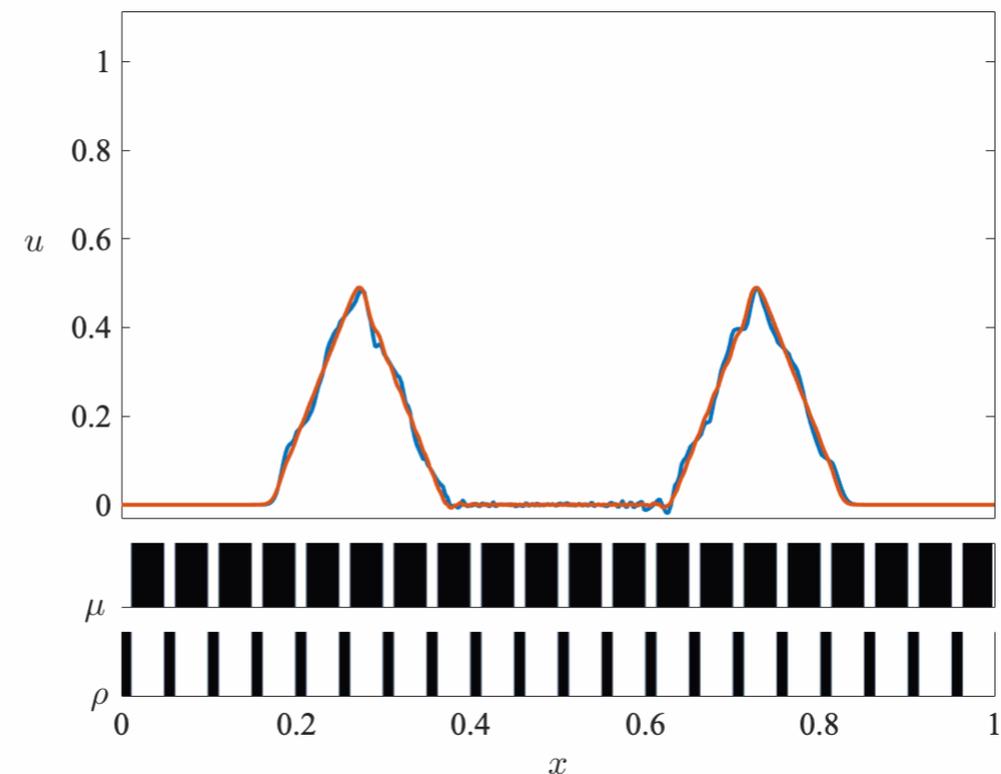
Homogenized equation (periodic):

$$\rho_{\text{eff}} \frac{\partial^2 u_0}{\partial t^2}(x, t) = \mu_{\text{eff}} \frac{\partial^2 u_0}{\partial x^2}(x, t)$$

$$\text{with } \rho_{\text{eff}} = \langle \rho \rangle \quad \text{and} \quad \mu_{\text{eff}} = \left\langle \frac{1}{\mu} \right\rangle^{-1}$$



using suitable hypotheses (*ergodicity*)

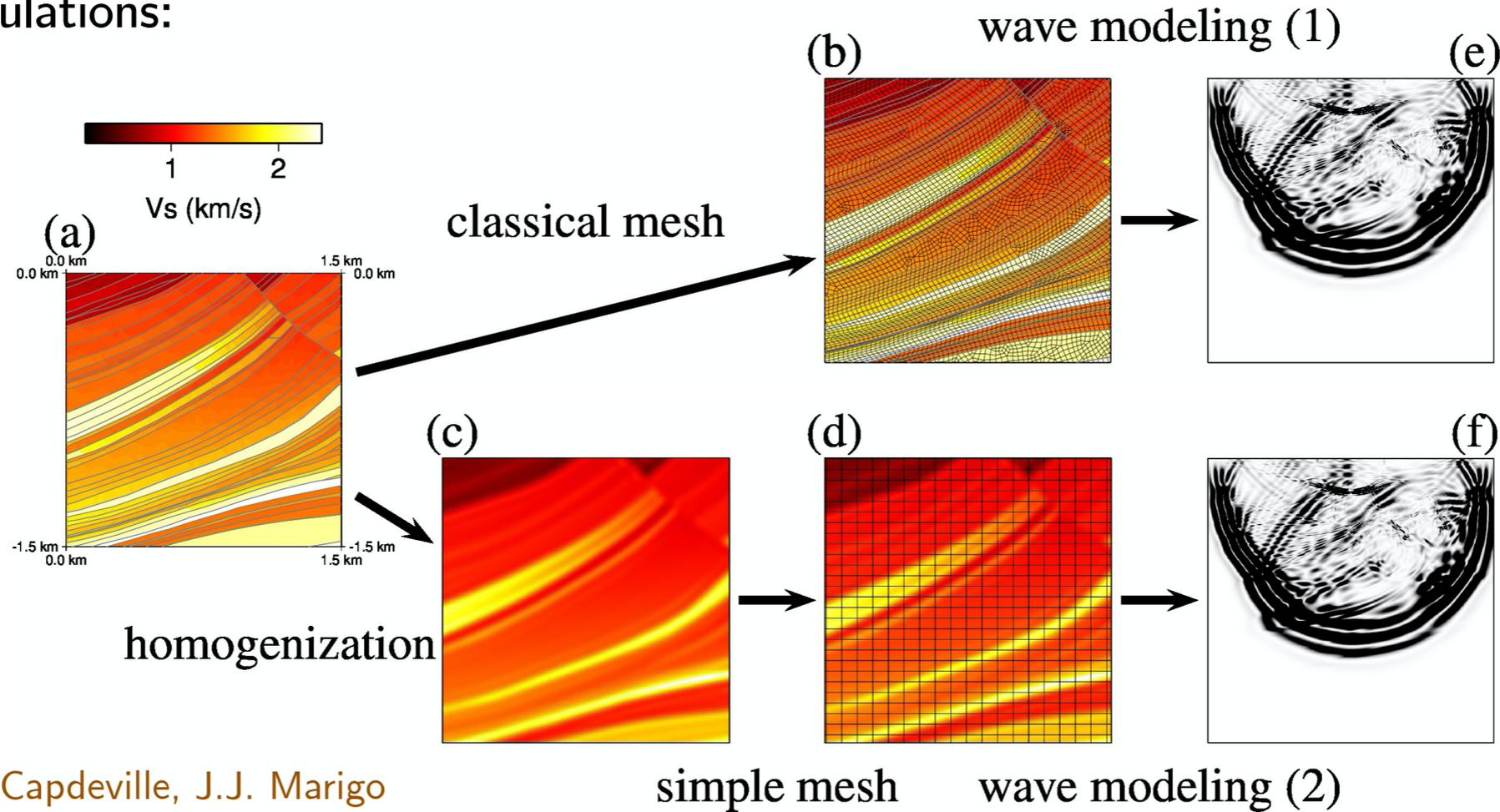


- ▶ Possible higher-order approximation, e.g.  $u = u_0 + \eta u_1 + O(\eta^2)$

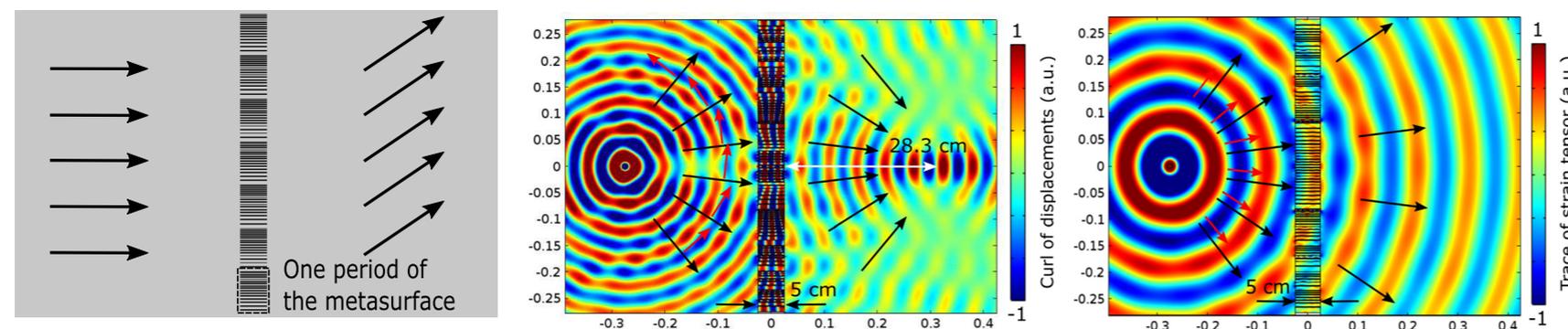
**!** Boundary conditions and localized sources require extra care and analysis!

# Motivations

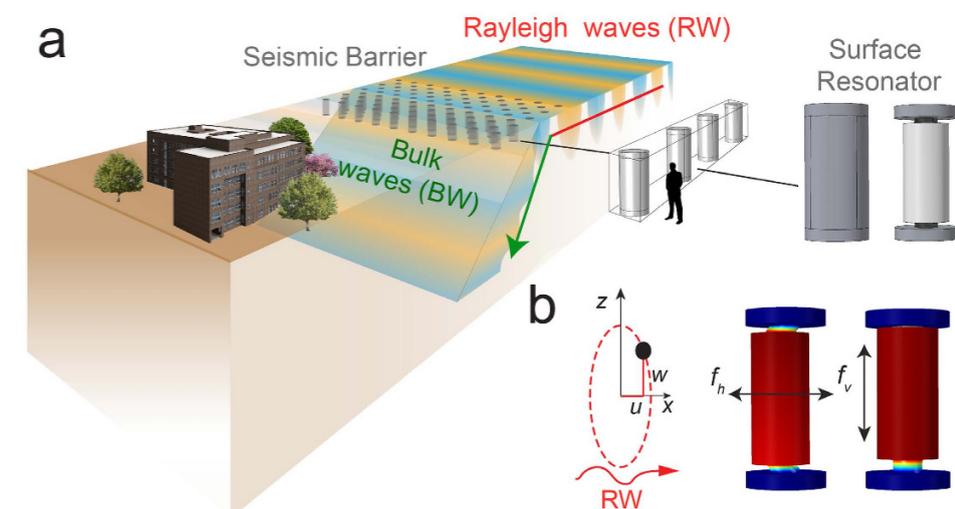
- ▶ Insight to essential phenomena and key parameters
- ▶ Efficient simulations:



- ▶ Manipulate waves: materials structured in bulk or surface



Su et al. 2018

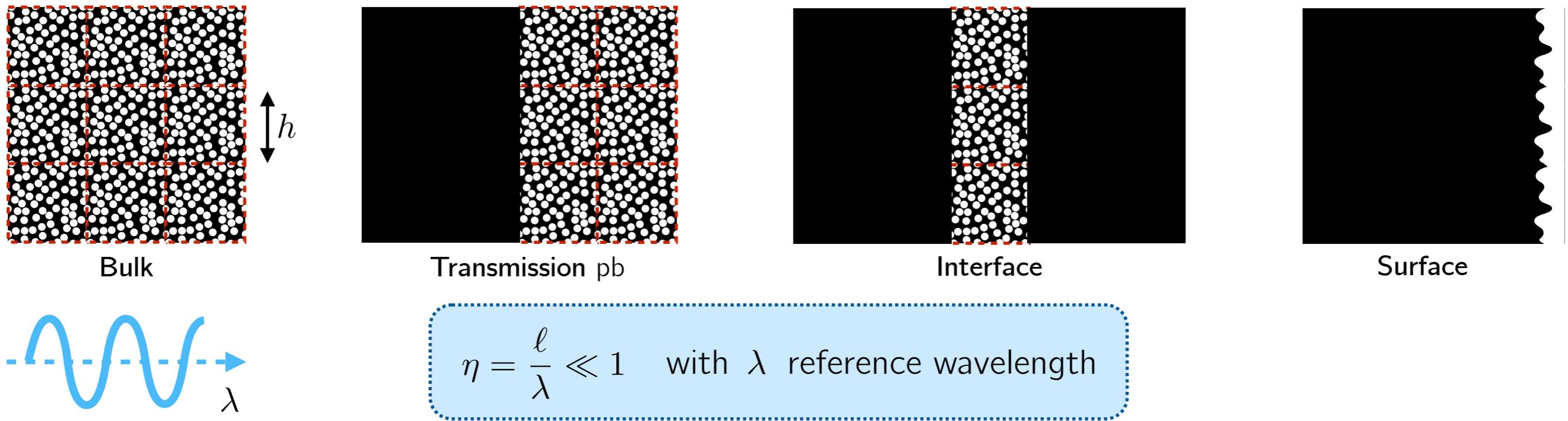


A. Palermo et al.

# Microstructured media

Willis, Sanchez-Palencia, Bensoussan-Lions-Papanicolaou, ...

- ▶ Different *geometrical* configurations of interest for waves: random, **periodic**, quasi-periodic



Recent references: Boutin, Craster, Guenneau, Marigo, Maurel, Pham, ...

- ▶ Different *material* configurations:

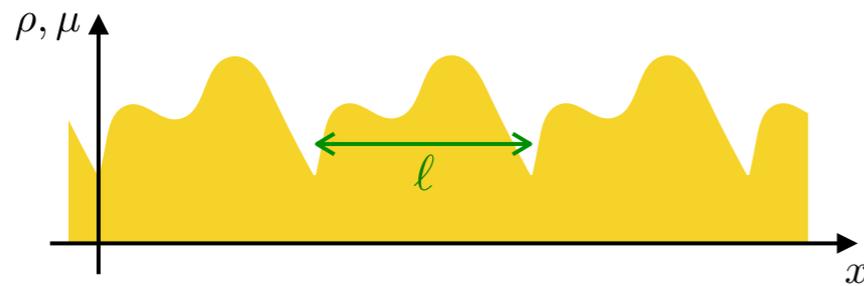
- Weak contrast :  $\rho(\mathbf{x}), \mathbf{L}(\mathbf{x})$  independent of  $\eta$   $\longrightarrow$  non-resonant case
- Strong contrast :  $\rho(\mathbf{x}), \mathbf{L}(\mathbf{x})$  depend on  $\eta$   $\longrightarrow$  resonant case

for example  $\frac{L_{\text{incl}}}{L_{\text{matrix}}} = \eta^\alpha$

1. Dispersion, high-order homogenization and design of microstructures
2. Homogenization with internal resonances
3. Dynamical homogenization with non-linearities (at interfaces)

# Dispersive behavior

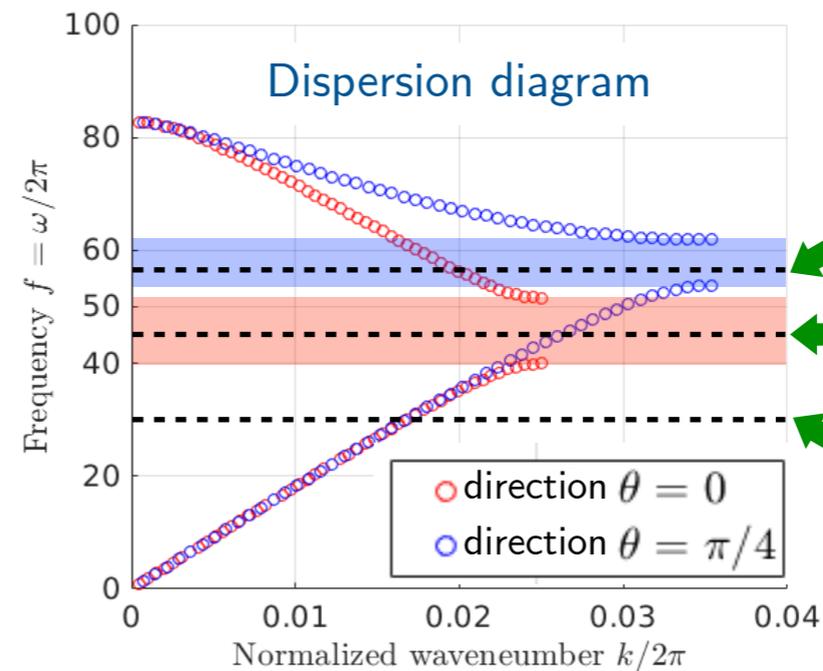
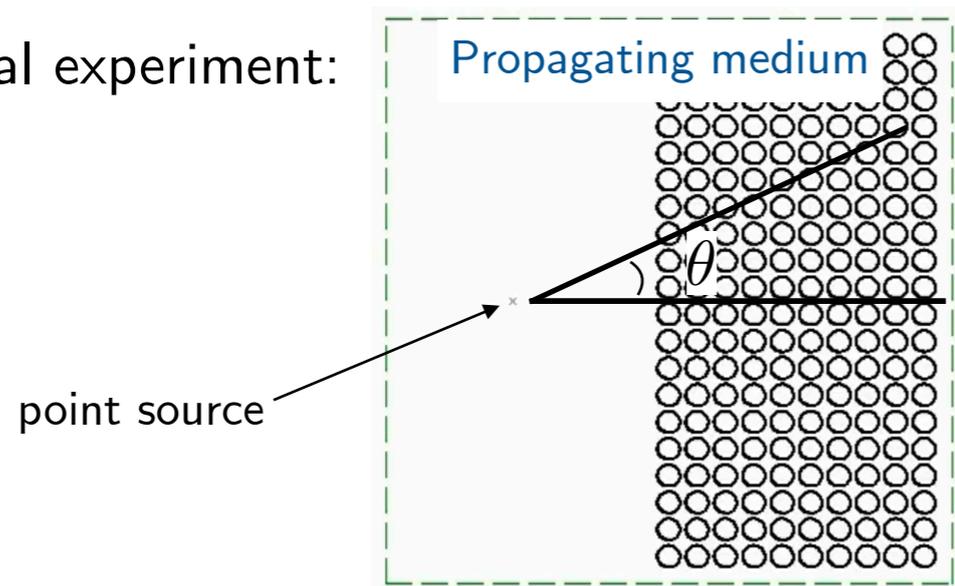
► In periodic media: **Floquet-Bloch theorem**



Eigenmodes of periodic wave equation in harmonic regime writes as  $\hat{U}(x) = \tilde{U}(x)e^{ikx}$  with  $\tilde{U}$   $\ell$ -periodic i.e.  $\tilde{U}(x) = \tilde{U}(x + \ell)$

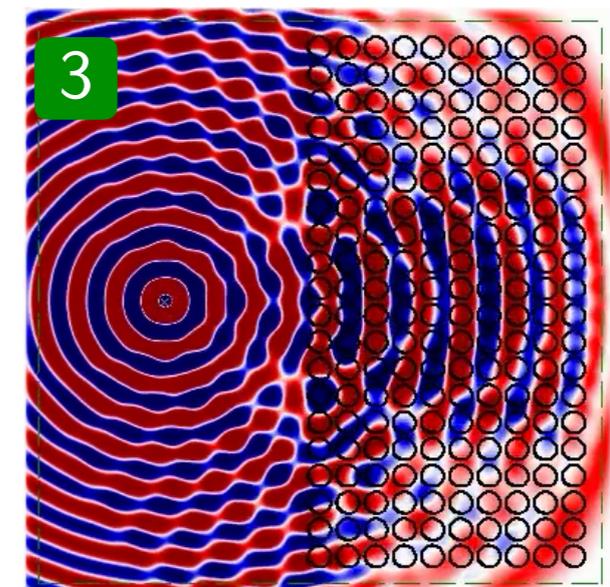
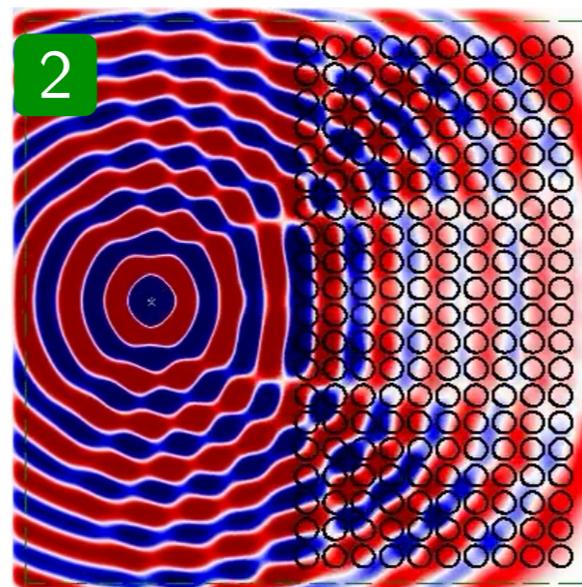
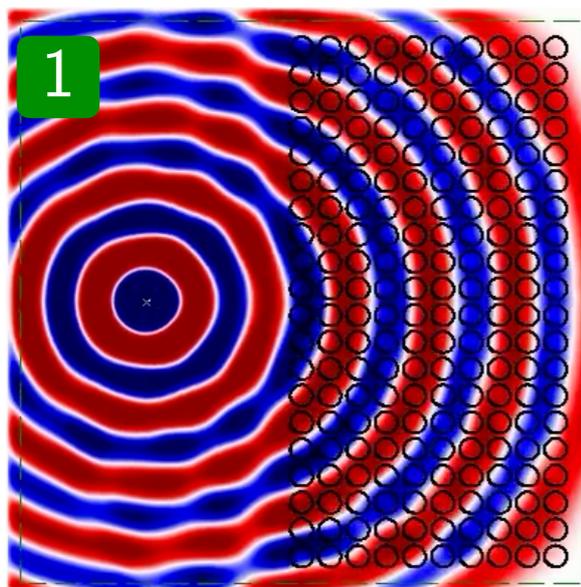
prediction of the waves properties **depending on propagation direction**  $k \in \mathbb{R}^d$

- Numerical experiment:



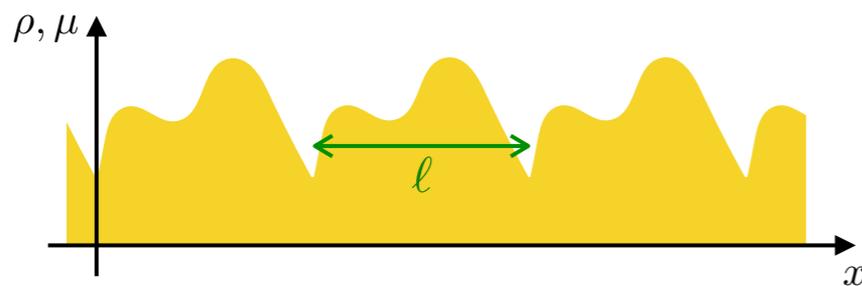
$$c(\omega) = \frac{\omega(k, \theta)}{k}$$

with  $k = \frac{2\pi}{\lambda}$



# Dispersive behavior

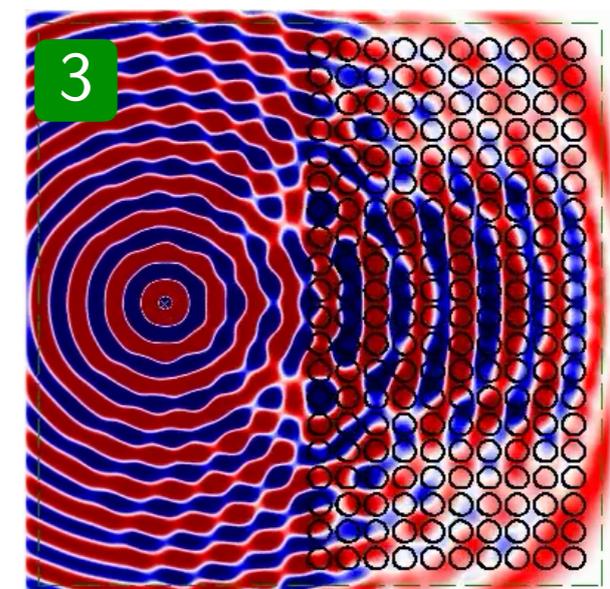
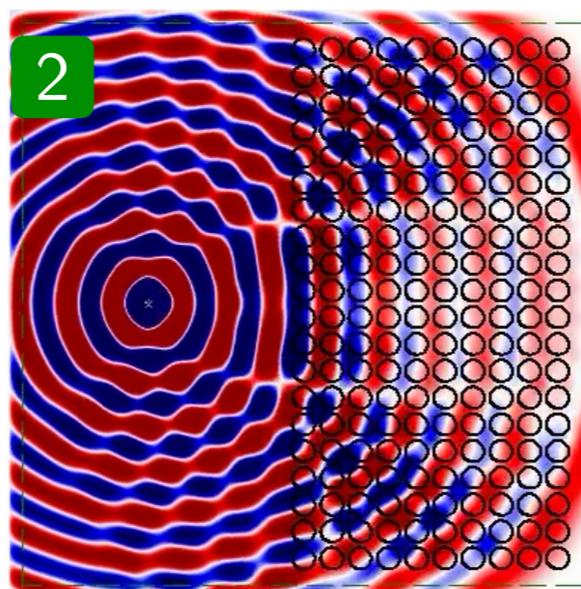
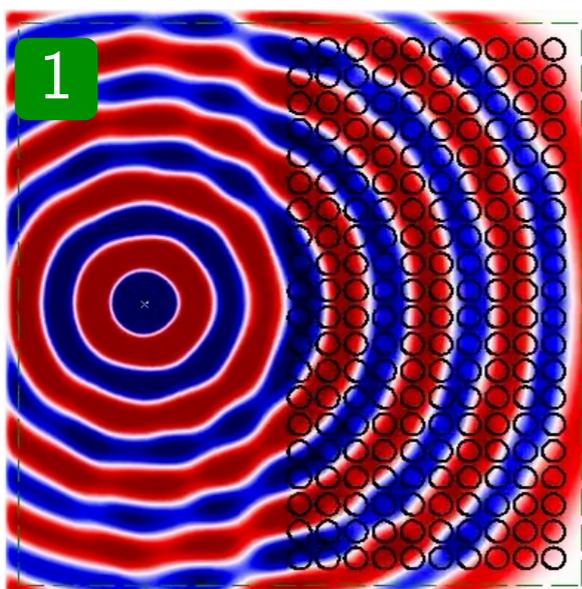
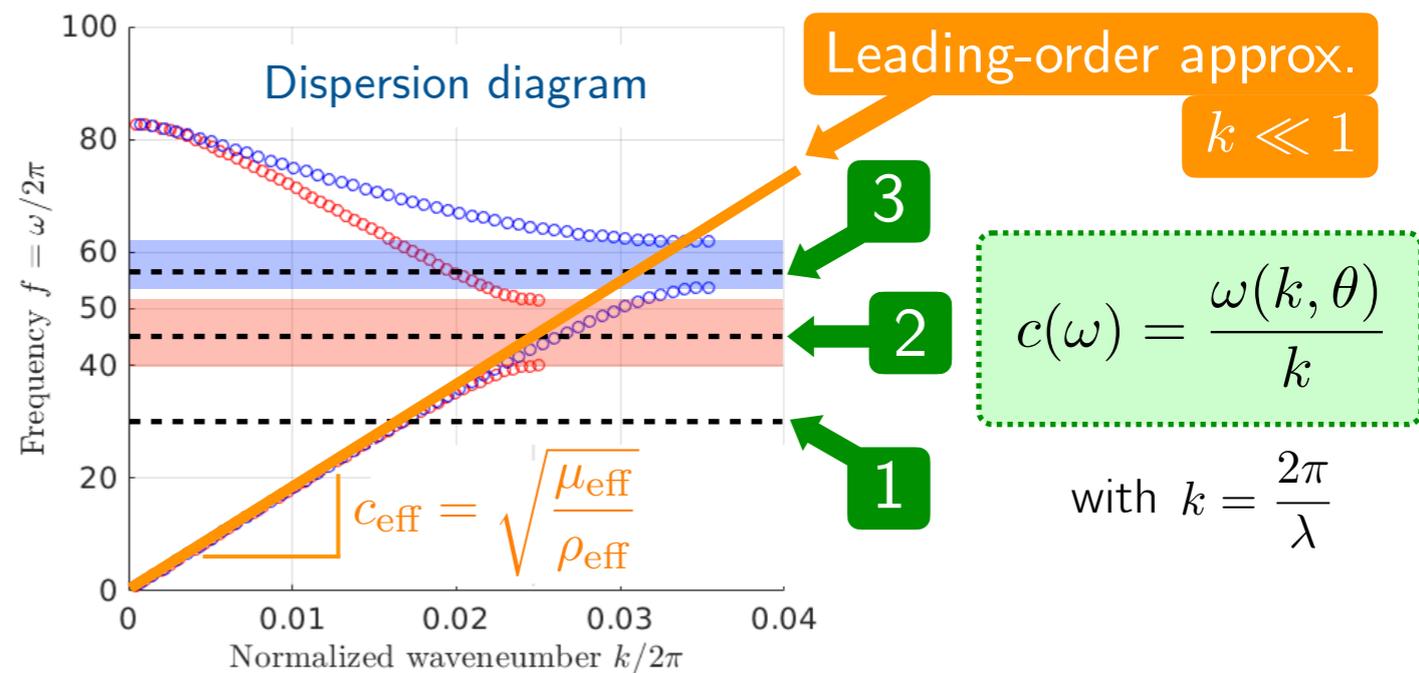
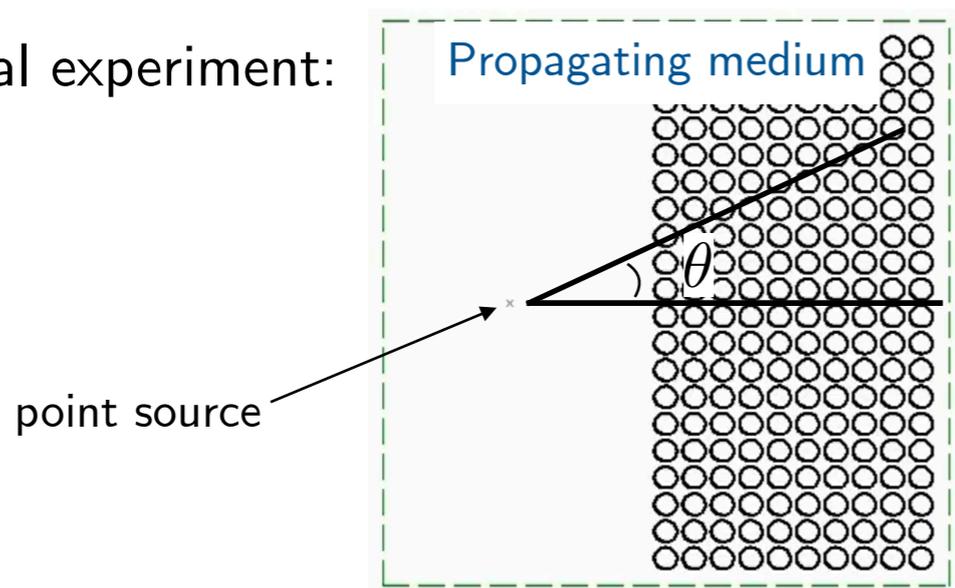
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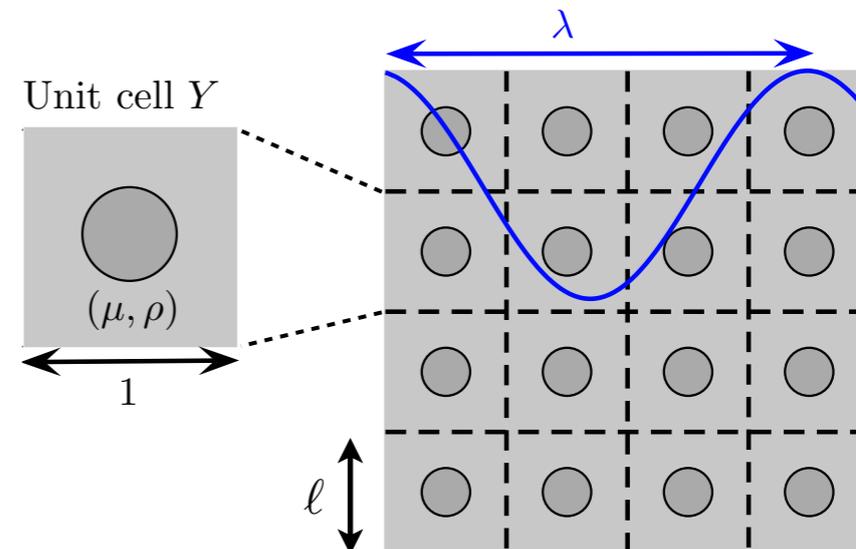


# Dynamical homogenization

- Scalar waves in time-harmonic regime:

$$\operatorname{div} \left( \mu \left( \frac{\mathbf{x}}{\ell} \right) \nabla u_\ell(\mathbf{x}) \right) + \rho \left( \frac{\mathbf{x}}{\ell} \right) \omega^2 u_\ell(\mathbf{x}) = 0$$

$\mathcal{C} = (\rho, \mu)$ :  $Y$ -periodic constitutive parameters  
(shear modulus, mass density)



- Long-wavelength regime  $\ell \ll \lambda$  :

two-scale asymptotic expansion of the solution

$$u_\ell(\mathbf{x}) = \underbrace{U(\mathbf{x})}_{\text{macroscopic field}} + \underbrace{\ell \mathbf{P}_1 \left( \frac{\mathbf{x}}{\ell} \right) \cdot \nabla U(\mathbf{x}) + \ell^2 \mathbf{P}_2 \left( \frac{\mathbf{x}}{\ell} \right) : \nabla^2 U(\mathbf{x}) + \dots}_{\text{oscillatory correctors with cell functions } P_j}$$

Bensoussan et al., Sanchez-Palencia, Auriault, Boutin, Andrianov, Guzina, ...

$$\begin{cases} \text{(i)} & \langle \mathbf{P} \rangle = \mathbf{0}, \quad \mathbf{P} \text{ periodic on } \partial Y, \\ \text{(ii)} & \mathbf{S}(\mathbf{y}) = \mu(\mathbf{y}) [\mathbf{E}(\mathbf{y}) + \nabla \mathbf{P}(\mathbf{y})], \\ \text{(iii)} & \operatorname{div} \mathbf{S}(\mathbf{y}) + \mathbf{f}(\mathbf{y}) = \mathbf{0}, \end{cases}$$

with prestrain  $\mathbf{E}$  and body force  $\mathbf{f}$

- Second-order (enriched) homogenized wave equation for  $U$ :

$$(\boldsymbol{\mu}_0 + \ell^2 \boldsymbol{\mu}_2 : \nabla^2) : \nabla^2 U + \omega^2 (\varrho_0 + \ell^2 \boldsymbol{\varrho}_2 : \nabla^2) U = 0$$

Effective parameters:  $\mathcal{C}_{\text{eff}}^{\text{dyn}} = (\varrho_0, \boldsymbol{\mu}_0, \boldsymbol{\varrho}_2, \boldsymbol{\mu}_2)$

$$\begin{cases} \varrho_0 = \langle \rho \rangle & \boldsymbol{\varrho}_2 = \langle \rho \mathbf{P}_2 \rangle \\ \boldsymbol{\mu}_0 = \langle \mu [\mathbf{I} + \nabla \mathbf{P}_1]^{\text{sym}} \rangle & \boldsymbol{\mu}_2 = \langle \mu [\mathbf{I} \otimes \mathbf{P}_2 + \nabla \mathbf{P}_3]^{\text{sym}} \rangle \end{cases}$$

# Effective properties and topological optimization

- ▶ **Effective dispersive effects:** plane wave solution  $U(\mathbf{x}) = e^{i(k\boldsymbol{\theta}\cdot\mathbf{x}-\omega t)} \Rightarrow$  dispersion relation  $\mathcal{D}(\omega, k, \boldsymbol{\theta}) = 0$
- ▶ **Phase velocity** for the effective model at wavenumber  $k$  and direction  $\boldsymbol{\theta}$ :

$$c(k, \boldsymbol{\theta}) \stackrel{\text{def}}{=} \frac{\omega(k, \boldsymbol{\theta})}{k} = c_0(\boldsymbol{\theta}) + \frac{1}{2} \frac{d(\boldsymbol{\theta})}{c_0(\boldsymbol{\theta})} (kl)^2 + o((kl)^2) \quad \text{with} \quad kl = \frac{\ell}{\lambda} \ll 1$$

$$c_0(\boldsymbol{\theta}) = \sqrt{\frac{\boldsymbol{\mu}_0}{\rho_0} : (\boldsymbol{\theta} \otimes \boldsymbol{\theta})}$$

low-frequency limit velocity

$$d(\boldsymbol{\theta}) = \left[ \frac{\boldsymbol{\rho}_2 \otimes \boldsymbol{\mu}_0 - \rho_0 \boldsymbol{\mu}_2}{\rho_0^2} \right]^{\text{sym}} :: (\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \boldsymbol{\theta})$$

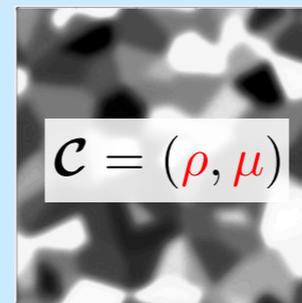
second-order dispersion term

## Question

How to design the microstructure so that it achieves target effective properties ?

 Allaire, Bendsøe, Sigmund, Yamada, Amstutz, ...

## ▶ Topological optimization problem:



$$\arg \min_{\mathbf{c}} \mathcal{F}(\mathbf{c}_{\text{eff}}^{\text{dyn}}(\mathbf{c}); \boldsymbol{\theta})$$

using  $\mathbf{c}_{\text{eff}}^{\text{dyn}} = (\rho_0, \boldsymbol{\mu}_0, \boldsymbol{\rho}_2, \boldsymbol{\mu}_2)$

▶ **Dependencies:**  $\mathbf{c} = (\rho, \boldsymbol{\mu}) \rightarrow \begin{matrix} \text{cell fct} \\ (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \end{matrix} \rightarrow \mathbf{c}_{\text{eff}}^{\text{dyn}} \rightarrow \begin{matrix} \text{macro obs.} \\ (c_0(\boldsymbol{\theta}), d(\boldsymbol{\theta})) \end{matrix} \rightarrow \text{cost fct } \mathcal{F}$

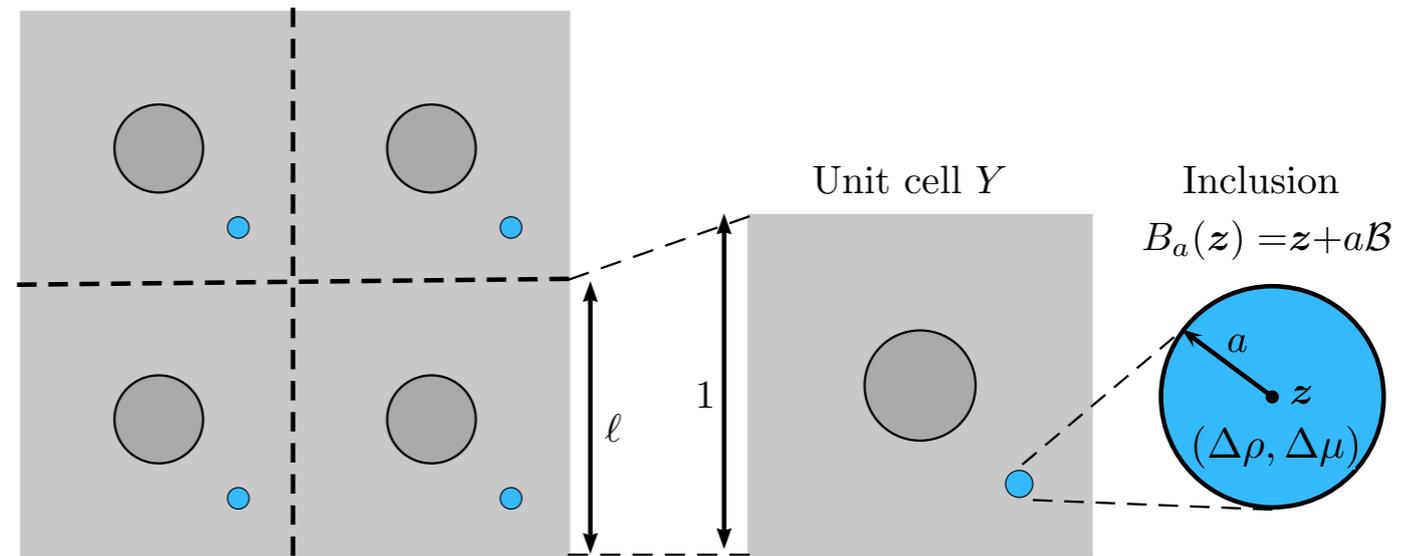
▶ **Example** of cost functional:  $\mathcal{F}(\mathbf{c}_{\text{eff}}^{\text{dyn}}(\mathbf{c}); \{\boldsymbol{\theta}^+, \boldsymbol{\theta}^-\}) = \frac{1}{2} \left( d(\boldsymbol{\theta}^-)^2 + \frac{1}{d(\boldsymbol{\theta}^+)^2} \right)$

▶ Possible constraints or parametrization  $(\rho, \boldsymbol{\mu})$

# Topological derivative

- ▶ Gradient-based approach:  
**Topological perturbation** of unit cell

$$\begin{array}{c}
 (\rho, \mu) \\
 \downarrow \\
 (\rho_a, \mu_a) = (\rho, \mu) + (\Delta\rho, \Delta\mu)\chi_{B_a}
 \end{array}$$



- ▶ Asymptotic expansion of cost functional:  $\mathcal{F}(\rho_a, \mu_a) \underset{a \rightarrow 0}{=} \mathcal{F}(\rho, \mu) + a^2 \mathcal{D}\mathcal{F} + o(a^2)$

with **topological derivative**  $\mathcal{D}\mathcal{F}(\rho, \mu; z, \mathcal{B}, \Delta\rho, \Delta\mu)$  Sokolowski, Garreau, Bonnet, Guzina, Amstutz, ...

- ▶ In the case of effective model  $\mathcal{F} \equiv \mathcal{F}(\mathbf{C}_{\text{eff}}^{\text{dyn}}(\rho, \mu); \boldsymbol{\theta})$  use chain rule:

$$\mathcal{D}\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \rho_0} \mathcal{D}\rho_0 + \frac{\partial \mathcal{F}}{\partial \mu_0} : \mathcal{D}\mu_0 + \frac{\partial \mathcal{F}}{\partial \rho_2} : \mathcal{D}\rho_2 + \frac{\partial \mathcal{F}}{\partial \mu_2} :: \mathcal{D}\mu_2 \quad \text{Bonnet, Cornaggia, Guzina}$$

- ▶ Computation of topological derivatives

of effective parameters  $(\mathcal{D}\rho_0, \mathcal{D}\mu_0, \mathcal{D}\rho_2, \mathcal{D}\mu_2)$

computation of 12 scalar static cell problems

$$\begin{cases}
 \langle \chi \rangle = 0, & \chi \text{ periodic on } \partial Y, \\
 \mathbf{S}(\mathbf{y}) = \mu(\mathbf{y}) [\mathbf{E}(\mathbf{y}) + \nabla \chi(\mathbf{y})], \\
 \text{div } \mathbf{S}(\mathbf{y}) + \mathbf{f}(\mathbf{y}) = \mathbf{0}
 \end{cases}$$

**Numerical strategies**



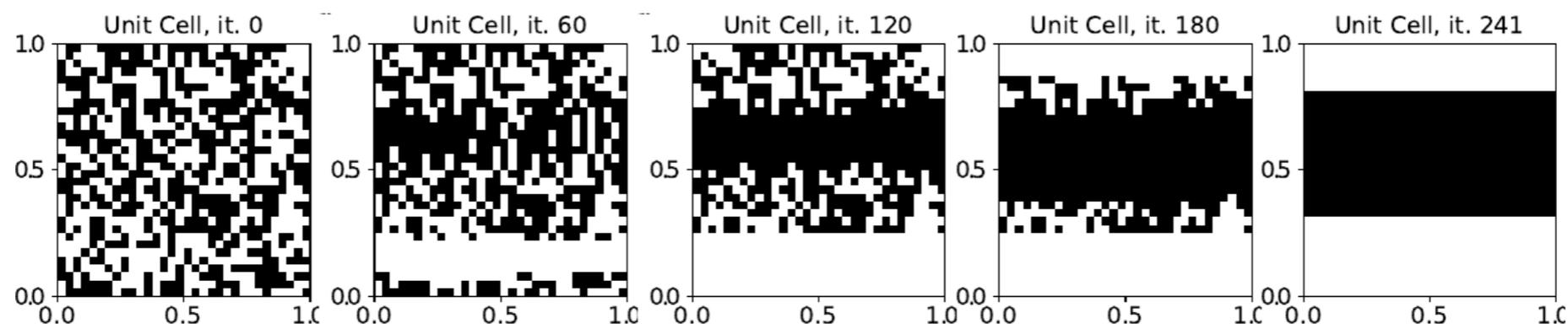
Finite elements methods, FFT-based methods, ...

Moulinec, Suquet, ...

# Material updating

- ▶ Example of cost functional:  $\mathcal{F}(\mathbf{c}_{\text{eff}}^{\text{dyn}}(\mathbf{c}); \{\theta^+, \theta^-\}) = \frac{1}{2} \left( d(\theta^-)^2 + \frac{1}{d(\theta^+)^2} \right)$ 
  - minimize dispersion at 0
  - maximize dispersion at  $\pi/2$
- ▶ Two-phase unit cell  $Y = Y_1 \cup Y_2$

→ Pixel-by-pixel algo: while  $(\min_{Y_1} \mathcal{DF} + \min_{Y_2} \mathcal{DF}) < 0$  swap phases at extremal pixels

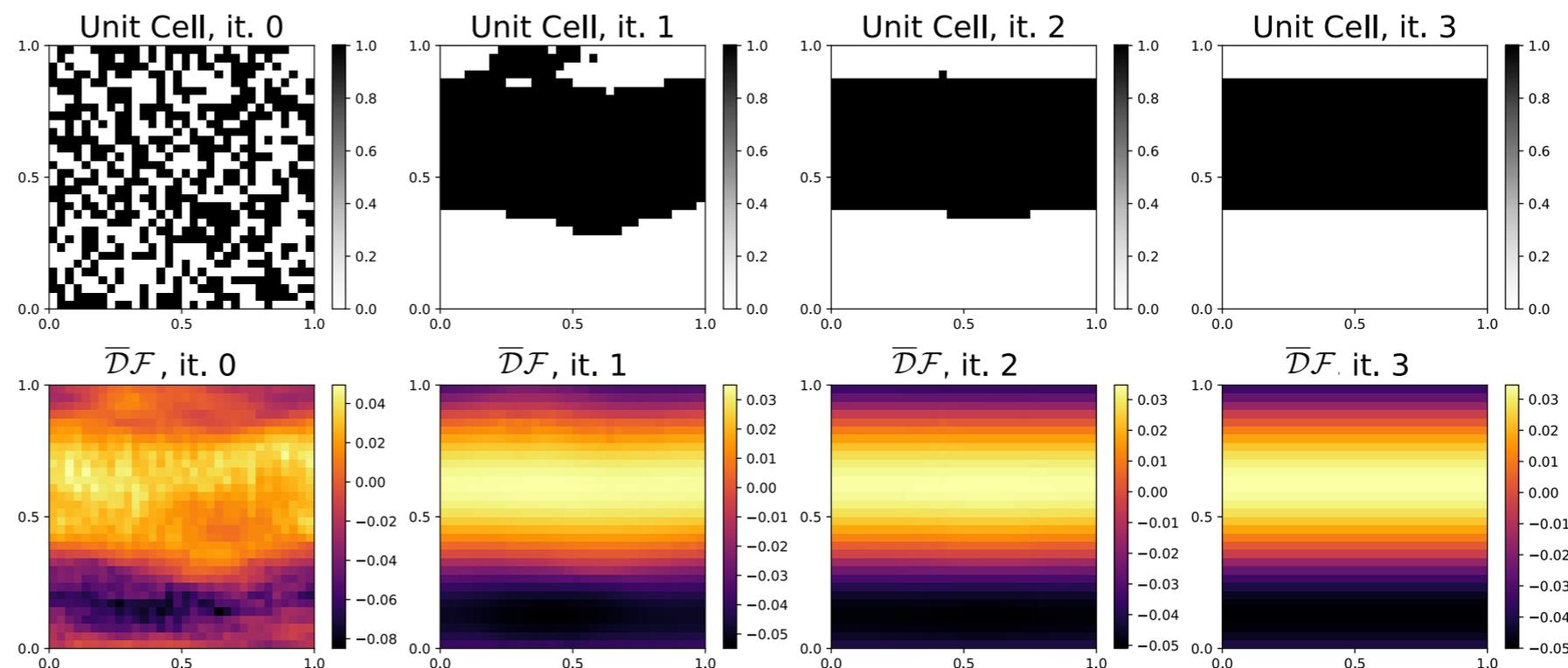


Level set algo:

$$\begin{cases} \psi(z) > 0 & \text{in } Y_1 \\ \psi(z) < 0 & \text{in } Y_2 \end{cases}$$

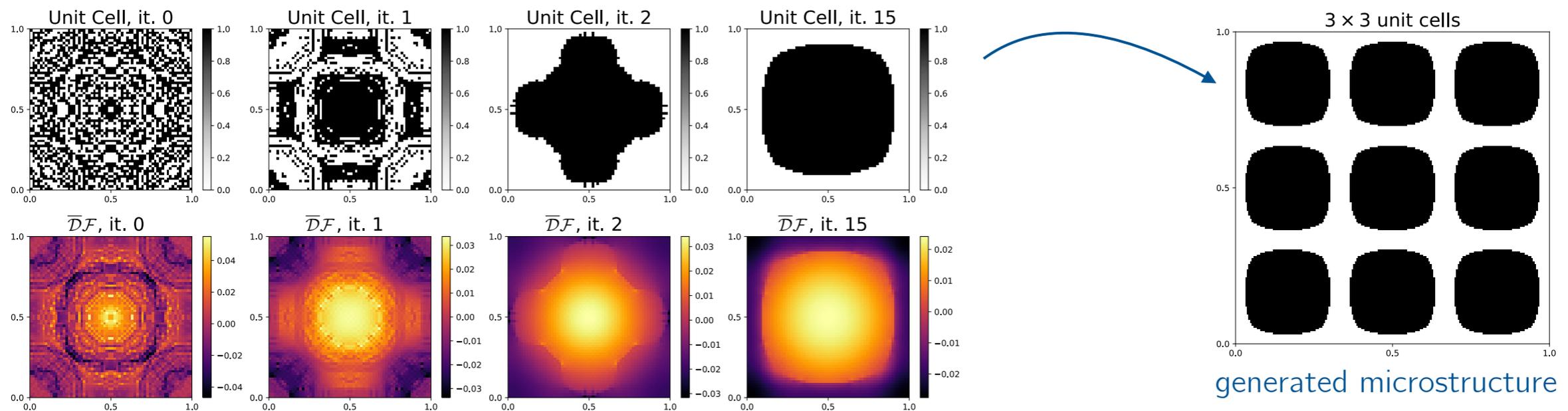
computed from  $\mathcal{DF}$

Amstutz, Andrä



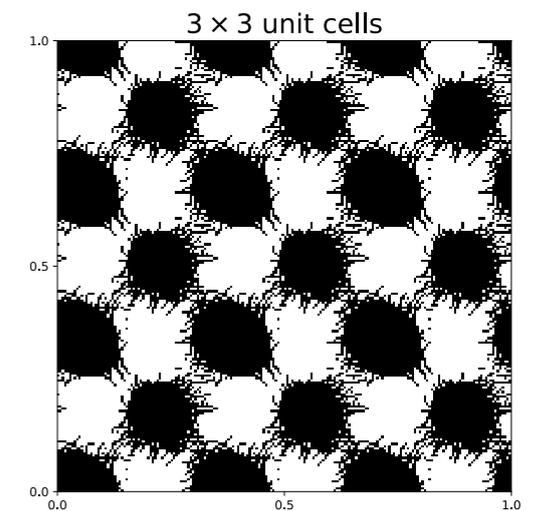
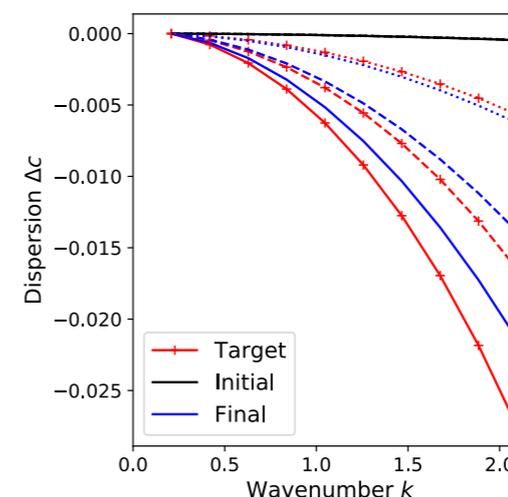
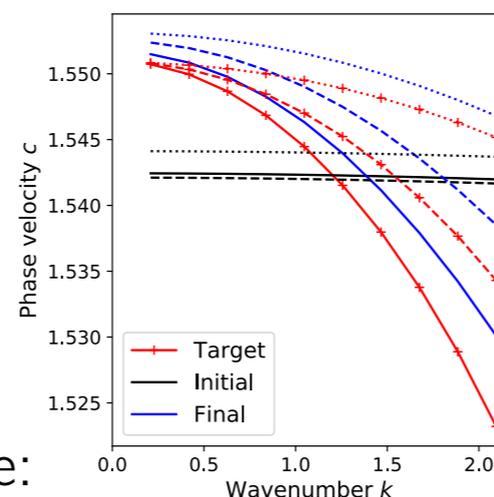
# Optimization and inversion results

- Optimization of dispersion w.r.t. multiple directions:  $\mathcal{F} = \frac{1}{2} \left( \sum_{j=1}^2 d(\theta_j^-)^2 + \sum_{j=3}^4 \frac{1}{d(\theta_j^+)^2} \right)$ 
  - minimize dispersion at  $\pm\pi/4$
  - maximize dispersion at  $0, \pi/2$



- Inversion: reconstruction of microstructure from dynamical meas.:  $\mathcal{F} = \sum_{k, \theta} |c_{\text{eff}}(k, \theta) - c_{\text{obs}}(k, \theta)|^2$

Objective: chessboard  
 synthetic data  $c_{\text{obs}}(k, \theta)$  generated  
 by Floquet-Bloch analysis



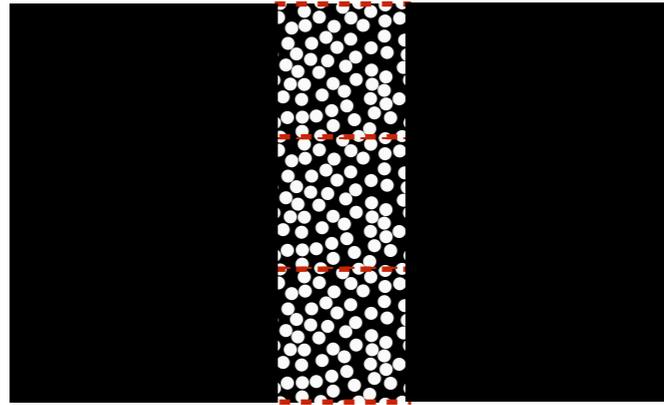
Identification from composite response:  
 see [Robin Valmalette poster!](#)

1. Dispersion, high-order homogenization and **design** of microstructures
2. Homogenization with **internal resonances**
3. Dynamical homogenization with **non-linearities** (at interfaces)

# Microstructured interface problem

► Context :

Interface pb



+

resonant case



$a$

effective jump conditions

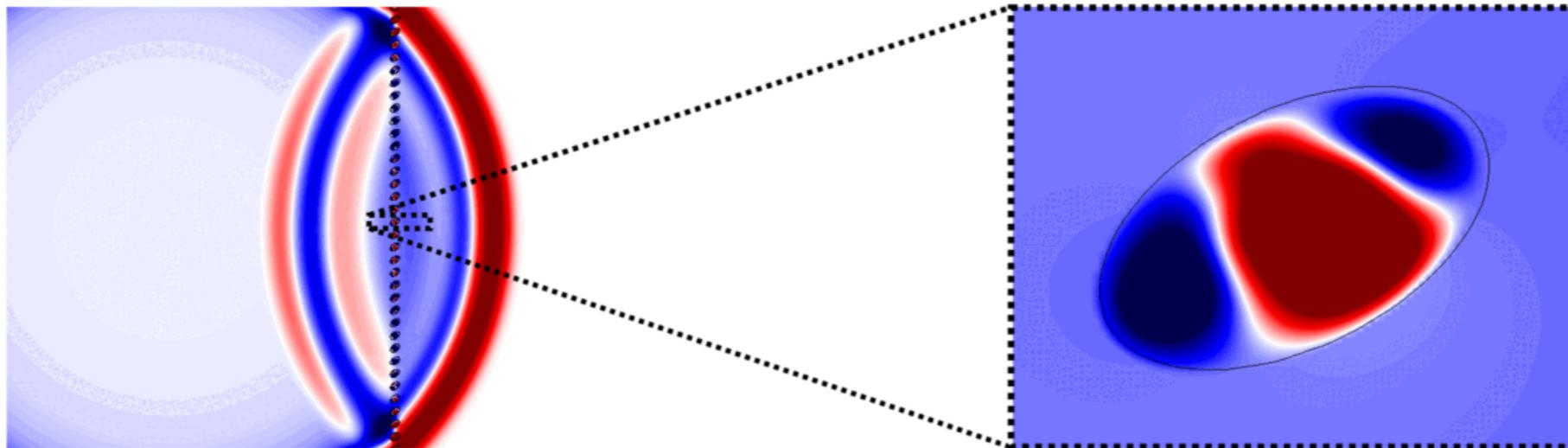
Delourme, Marigo, Maurel, Pham,...

- 2D scalar waves (e.g. anti-plane elasticity):

$$\begin{cases} \frac{\partial \Sigma_h}{\partial t} = \mu_h(\mathbf{X}) \nabla V_h \\ \rho_h(\mathbf{X}) \frac{\partial V_h}{\partial t} = \nabla \cdot \Sigma_h \end{cases}$$

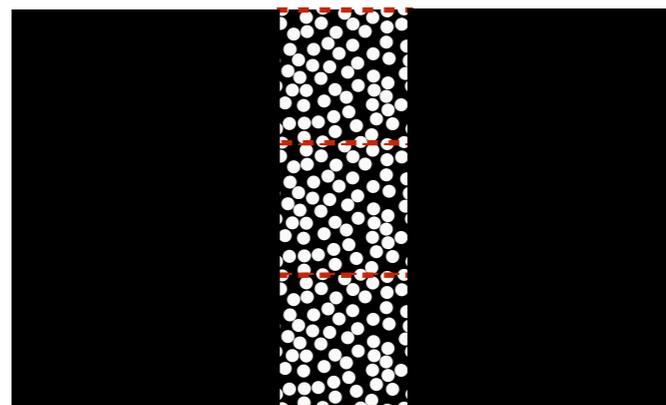
$$\rho_i \approx \rho_m \text{ and } \mu_i \approx \eta^2 \mu_m$$

- behavior in the time-domain



# Time-domain homogenization

► Context :



Interface pb



resonant case



effective jump conditions

📄 Delourme, Marigo, Maurel, Pham,...

- 2D scalar waves (e.g. anti-plane elasticity):

$$\begin{cases} \frac{\partial \Sigma_h}{\partial t} = \mu_h(\mathbf{X}) \nabla V_h \\ \rho_h(\mathbf{X}) \frac{\partial V_h}{\partial t} = \nabla \cdot \Sigma_h \end{cases}$$

$$\rho_i \approx \rho_m \text{ and } \mu_i \approx \eta^2 \mu_m$$

- asymptotic expansion of the solution :  $V = V_0 + hV_1, \Sigma = \Sigma_0 + h\Sigma_1$

➡ effective jump conditions (freq.)

$$\begin{cases} [[\hat{V}]]_a = \mathcal{B} \cdot \langle \nabla \hat{V} \rangle_a \\ [[\hat{\Sigma}_1]]_a = \mathcal{C} : \langle \nabla \hat{\Sigma} \rangle_a + h \hat{\mathcal{D}}(\omega) \langle \text{div } \hat{\Sigma}(\omega) \rangle_a \end{cases}$$

$$\text{with } \hat{\mathcal{D}}(\omega) = \alpha_0 - \sum_{r=1}^{\infty} \alpha_r^2 \frac{\omega^2}{\omega^2 - \omega_r^2}$$

📄 M. Touboul, PhD 2021

- homogenization in the **time-domain**

non-local terms:  $h \mathcal{D} *_{t} \langle \text{div } \Sigma \rangle_a$

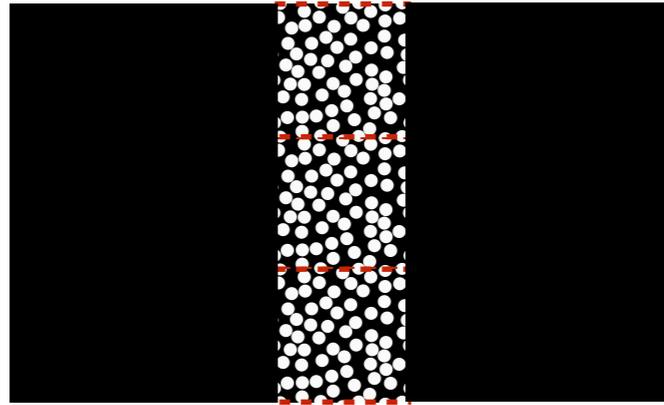
- development of simulation tools

- optimization of microstructures (non-resonant case) with R. Cornaggia

# Effective model

► Context :

Interface pb



+

resonant case

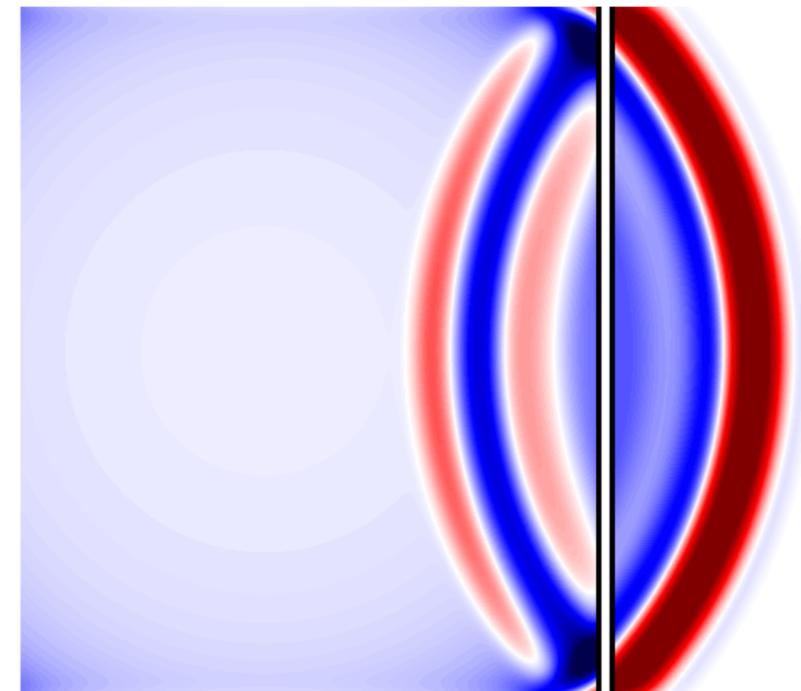
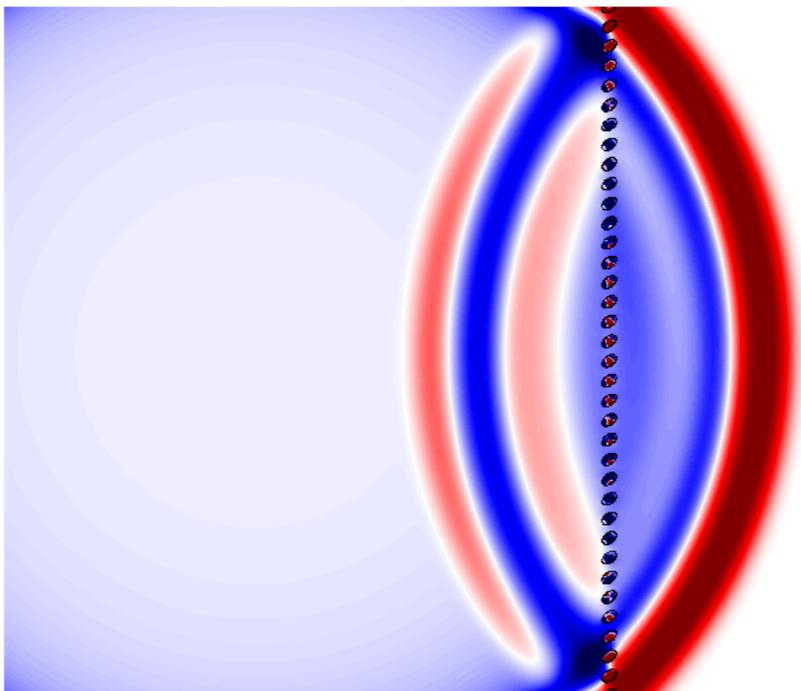


$a$

effective jump conditions

- homogenized model (approx. 1<sup>st</sup> order) : conservation eq. + jump cond.

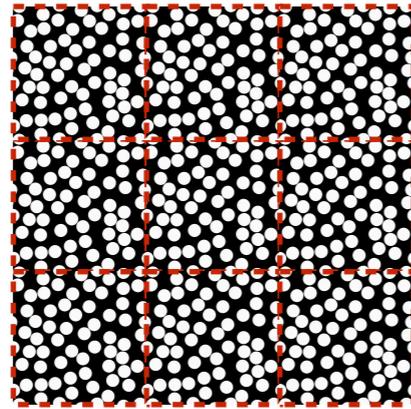
$$\begin{cases} \frac{\partial \Sigma}{\partial t} = \mu_m \nabla V \\ \rho_m \frac{\partial V}{\partial t} = \nabla \cdot \Sigma \end{cases} \quad \begin{cases} [[V]]_a = \mathcal{B} \cdot \langle \nabla V \rangle_a \\ [[\Sigma_1]]_a = \mathcal{S} \langle \frac{\partial \Sigma_1}{\partial X_1} \rangle_a + \mathcal{C} \cdot \langle \nabla \Sigma_2 \rangle_a + h \mathcal{D} *_t \langle \text{div } \Sigma \rangle_a \end{cases}$$



# Resonant effective models

► Context :

Bulk pb



+

Resonant case



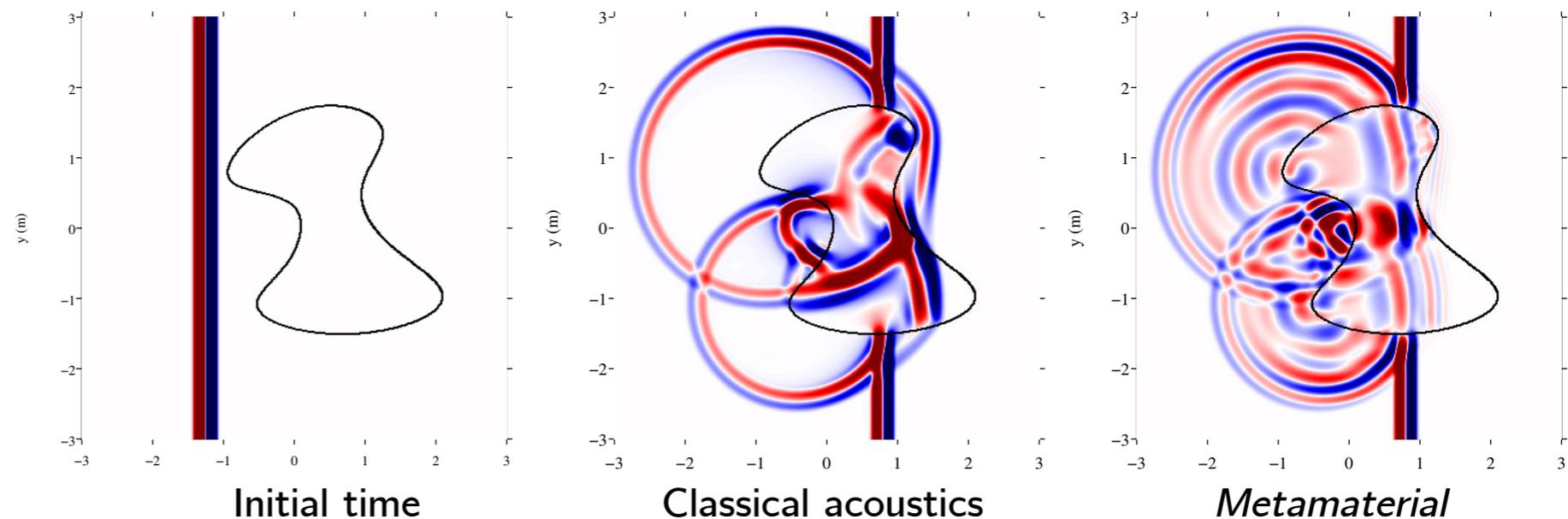
$\rho_{\text{eff}}(\omega), \mathbf{L}_{\text{eff}}(\omega)$

► Tools : effective constitutive relations. Example in acoustics:  $(\mathbf{v}, p) = (\text{velocity}, \text{pressure})$

$$\begin{cases} \rho_a \frac{\partial \mathbf{v}}{\partial t} + \nabla p = \rho_a \mathbf{g}, \\ \kappa_a^{-1} \frac{\partial p}{\partial t} + \text{div } \mathbf{v} = \kappa_a^{-1} f, \end{cases} \quad \text{with} \quad \begin{aligned} \rho_a &\longleftarrow \hat{\rho}(\omega) = \rho_a \left( 1 - \frac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2} \right), \\ \kappa_a^{-1} &\longleftarrow \hat{\kappa}^{-1}(\omega) = \kappa_a^{-1} \left( 1 - \frac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2 - i\gamma\omega} \right). \end{aligned}$$

► Objectives : time-domain simulations (frequency-dependent law = non-locality in time)

- Augmented formulation (introduction of *auxiliary fields*)
- Hyperbolic syst. of order 1
- Well-posedness in time
- Dispersive properties of media
- Standard numerical schemes

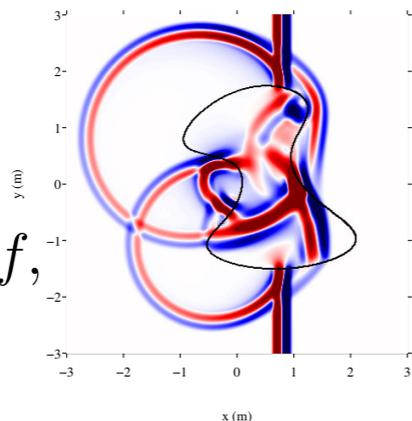


# Handling resonant effective models

- Non-local (in time) effective constitutive relations

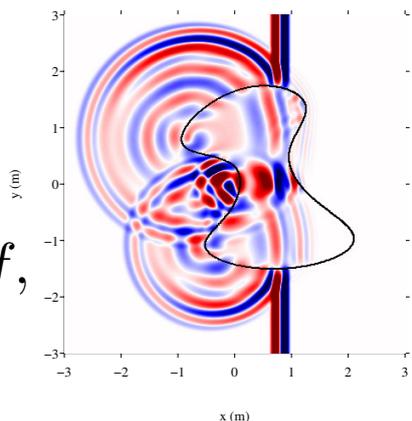
$(\mathbf{v}, p) = (\text{velocity}, \text{pressure})$

$$\begin{cases} \rho_a \frac{\partial \mathbf{v}}{\partial t} + \nabla p = \rho_a \mathbf{g}, \\ \kappa_a^{-1} \frac{\partial p}{\partial t} + \text{div } \mathbf{v} = \kappa_a^{-1} f, \end{cases}$$



internal resonances

$$\begin{cases} \rho * \frac{\partial \mathbf{v}}{\partial t} + \nabla p = \rho_a \mathbf{g}, \\ \kappa^{-1} * \frac{\partial p}{\partial t} + \text{div } \mathbf{v} = \kappa_a^{-1} f, \end{cases}$$



- History-dependent behavior: memory intensive if past values of fields are stored

- Auxiliary variables trick (a.k.a. internal variables)

Given  $\mathcal{F} \left[ \rho * \frac{\partial \mathbf{v}}{\partial t} \right] = \hat{\rho}(\omega) i\omega \hat{\mathbf{v}}(\omega)$  with  $\hat{\rho}(\omega) = \rho_a \left( 1 - \frac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2} \right)$

Define auxiliary field  $\hat{\mathbf{w}}(\omega) = \left( \frac{-1}{\omega^2 - \omega_\rho^2} \right) \hat{\mathbf{v}}(\omega)$  so that:  $\rho * \frac{\partial \mathbf{v}}{\partial t} + \dots$

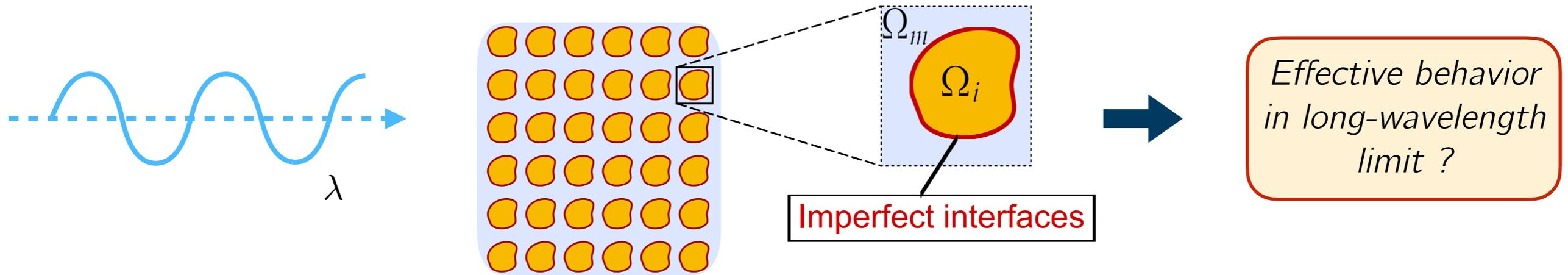
augmented formulation  
overall energy conservation

frequency dependence transferred to auxiliary field

$$\begin{cases} \rho_a \frac{\partial \mathbf{v}(t)}{\partial t} + \rho_a \Omega_\rho^2 \frac{\partial \mathbf{w}}{\partial t}(t) + \dots \\ \frac{\partial^2 \mathbf{w}(t)}{\partial t^2} + \omega_\rho^2 \mathbf{w}(t) = \mathbf{v}(t) \quad \text{linear ODE} \end{cases}$$

1. Dispersion, high-order homogenization and **design** of microstructures
2. Homogenization with **internal resonances**
3. Dynamical homogenization with **non-linearities** (at interfaces)

# Imperfect interfaces



- Jump conditions to model imperfect transmission conditions between matrix and inclusions

Rational approach through homogenization of **thin interphases**

$$\begin{cases} [[U]]_a = B_1 \langle \boldsymbol{\Sigma} \cdot \mathbf{n} \rangle_a + B_2 \left\langle \frac{\partial U}{\partial X_2} \right\rangle_a & \text{displacement} \\ [[\boldsymbol{\Sigma} \cdot \mathbf{n}]]_a = S \left\langle \frac{\partial^2 U}{\partial t^2} \right\rangle_a + \mathbf{C} \cdot \left\langle \frac{\partial \boldsymbol{\Sigma}}{\partial X_2} \right\rangle_a & \text{stress} \end{cases}$$

- Limit behavior:  $B_2 = \mathbf{C} = 0 \longrightarrow$  **spring-mass jump conditions**

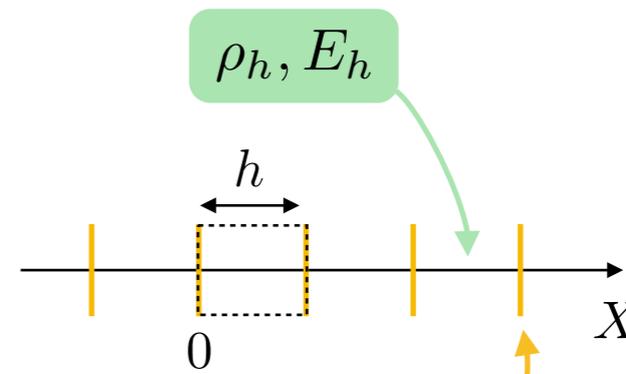
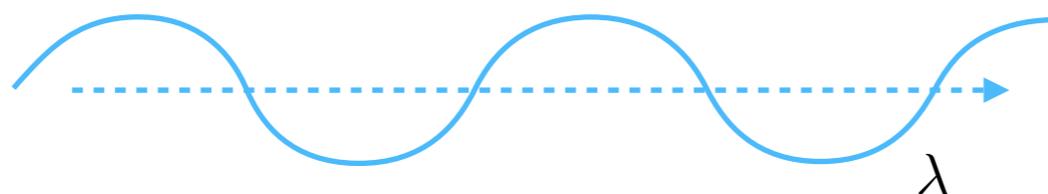
$$\begin{cases} \mathbf{K} \cdot [[U]]_{\mathbf{x}} = \langle \boldsymbol{\Sigma} \cdot \mathbf{n} \rangle_{\mathbf{x}} & \mathbf{K} : \text{stiffness} \\ [[\boldsymbol{\Sigma} \cdot \mathbf{n}]]_{\mathbf{x}} = \mathbf{M} \cdot \left\langle \frac{\partial^2 U}{\partial t^2} \right\rangle_{\mathbf{x}} & \mathbf{M} : \text{mass} \end{cases}$$

together with phenomenological models

 Jones, Lebon, Licht, Rizzoni, Sevostianov, Tattersall, Whittier,...

# Microstructured problem

- ▶ 1D array of non-linear imperfect interfaces



- wave equation:

$$\rho_h(X) \frac{\partial^2 U_h}{\partial t^2}(X, t) = \frac{\partial \Sigma_h}{\partial X}(X, t) + F(X, t)$$

- linear stress-strain relation:

$$\Sigma_h(X, t) = E_h(X) \frac{\partial U_h}{\partial X}(X, t)$$

- interface conditions:

$$\begin{cases} M \left\langle\left\langle \frac{\partial^2 U_h}{\partial t^2}(\cdot, t) \right\rangle\right\rangle_{X_n} = \llbracket \Sigma_h(\cdot, t) \rrbracket_{X_n} \\ \left\langle\left\langle \Sigma_h(\cdot, t) \right\rangle\right\rangle_{X_n} = K \mathcal{R}(\llbracket U_h(\cdot, t) \rrbracket_{X_n}) \end{cases}$$

$M, K$  : interface mass and stiffness

▶ jump and mean operators:  $\llbracket g \rrbracket_{X_n} = g(X_n^+) - g(X_n^-)$  and  $\left\langle\left\langle g \right\rangle\right\rangle_{X_n} = \frac{1}{2} (g(X_n^+) + g(X_n^-))$

▶ Setting of interest:

- 1  $\left\{ \begin{array}{l} \text{- long-wavelength regime } h \ll \lambda \\ \text{- non-linear interface behavior } \mathcal{R} \end{array} \right.$
- 2 from observations  $\left\{ \begin{array}{l} M/h \sim \frac{1}{h} \int_0^h \rho_h \, dX \\ Kh \sim \left( \frac{1}{h} \int_0^h E_h^{-1} \, dX \right)^{-1} \end{array} \right.$

▶ Objective: effective dynamical model at the 1<sup>st</sup>-order, i.e. approximation  $U_h(X, t) = U^{(1)}(X, t) + o(h)$

# Constitutive interface law

► Constitutive assumptions:

1)  $M \geq 0$  and  $K > 0$

2)  $\mathcal{R} : (-d, +\infty) \rightarrow \mathbb{R}$  smooth

maximum compressibility length:  $d \in \mathbb{R}_+ \cup \{+\infty\}$

technical assumptions:  $\mathcal{R}(0) = 0, \mathcal{R}' > 0$

$$\mathcal{R}'' < 0 \text{ or } \mathcal{R}'' = 0$$

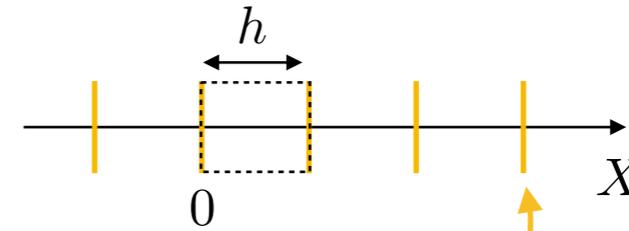
► Phenomenological models

 Achenbach, Norris, Bandis, Sevostianov, Broda, ...

► Examples:

model degenerates to

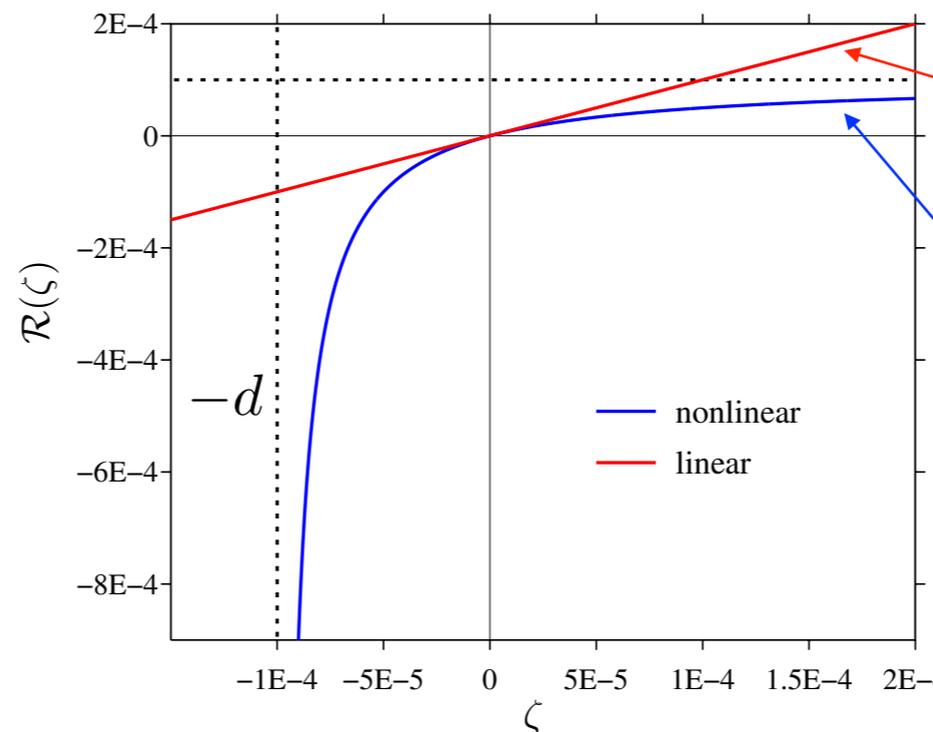
- perfect transmission cond.
- linear law
- unilateral contact
- ...



- interface conditions:

$$\begin{cases} M \left\langle\left\langle \frac{\partial^2 U_h(\cdot, t)}{\partial t^2}(\cdot, t) \right\rangle\right\rangle_{X_n} = [\Sigma_h(\cdot, t)]_{X_n} \\ \left\langle\left\langle \Sigma_h(\cdot, t) \right\rangle\right\rangle_{X_n} = K \mathcal{R}([\![U_h(\cdot, t)]\!]_{X_n}) \end{cases}$$

$M, K$ : interface mass and stiffness



$\mathcal{R}(\zeta) = \zeta$  linear (spring-like)

$\mathcal{R}(\zeta) = \frac{\zeta}{1 + \zeta/d}$  hyperbolic

# Microstructure behavior

► For a **single** interface:

- existence and uniqueness of a solution to elastodyn. equations  Junca, Lombard
- generation of harmonics
- amplitude of harmonics increases with source amplitude and non-monotonic behavior w.r.t. freq.

► **Energy analysis in array:**  $\mathcal{E}_h = \mathcal{E}_h^m + \mathcal{E}_h^i$  such that  $\frac{d}{dt}\mathcal{E}_h = 0$  without source

bulk mech. energy

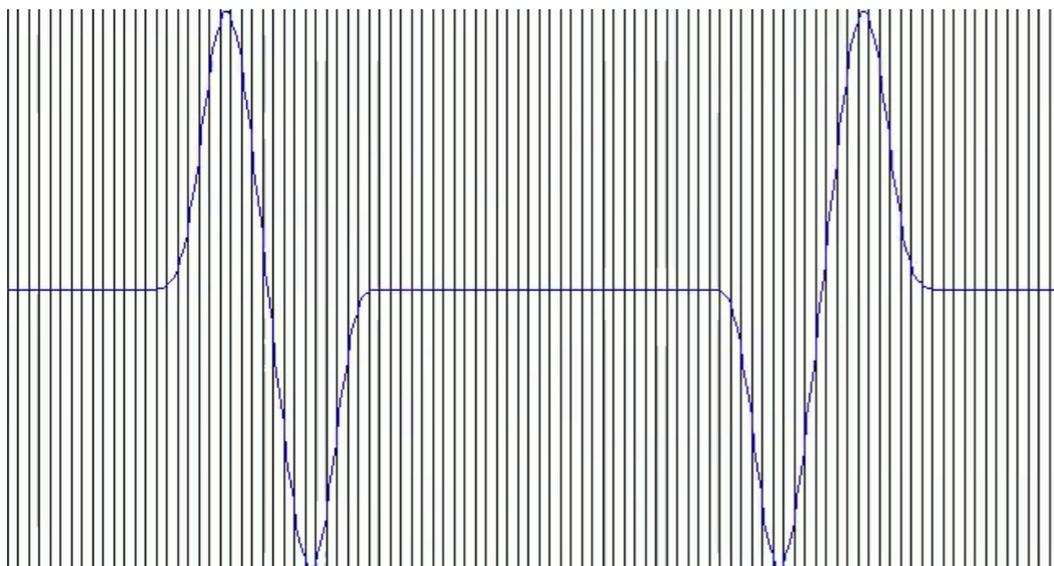
interface energy

$$\mathcal{E}_h^m(t) = \frac{1}{2} \int_I \left\{ \rho_h(X) V_h(X, t)^2 + \frac{1}{E_h(X)} \Sigma_h(X, t)^2 \right\} dX$$

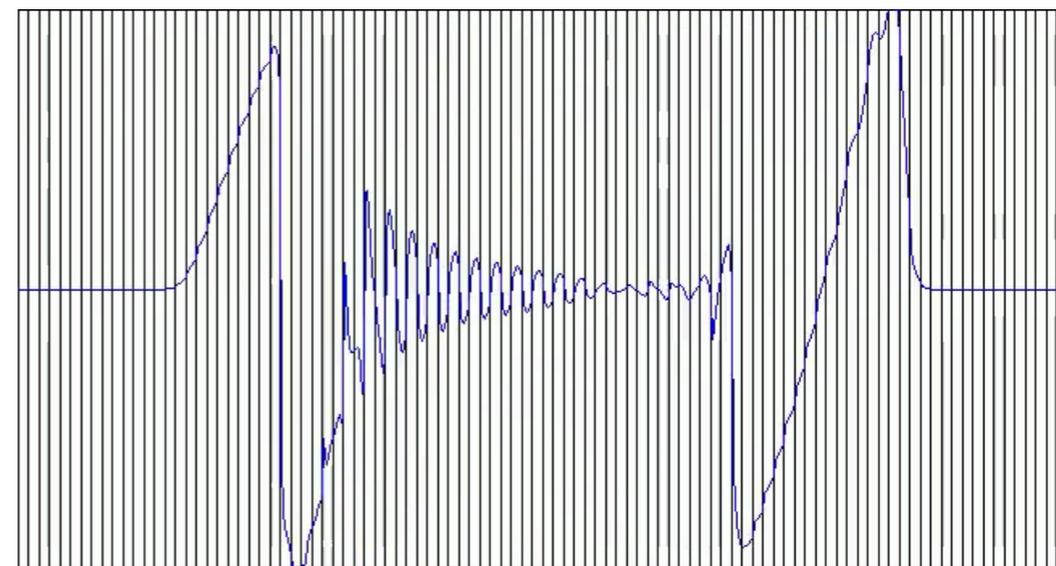
$$\mathcal{E}_h^i(t) = \sum_{X_n^I} \left\{ \frac{1}{2} M \langle\langle V_h(\cdot, t) \rangle\rangle_{X_n^I}^2 + K \int_0^{\mathcal{R}^{-1}(\langle\langle \Sigma_h(\cdot, t) \rangle\rangle_{X_n^I} / K)} \mathcal{R}(\zeta) d\zeta \right\}$$

► **Numerical simulation:** source with amplitude  $A$  and *fixed* central frequency

$$A = 0.1$$



$$A = 120$$



# Microstructure behavior

► For a **single** interface:

- existence and uniqueness of a solution to elastodyn. equations  Junca, Lombard
- generation of harmonics
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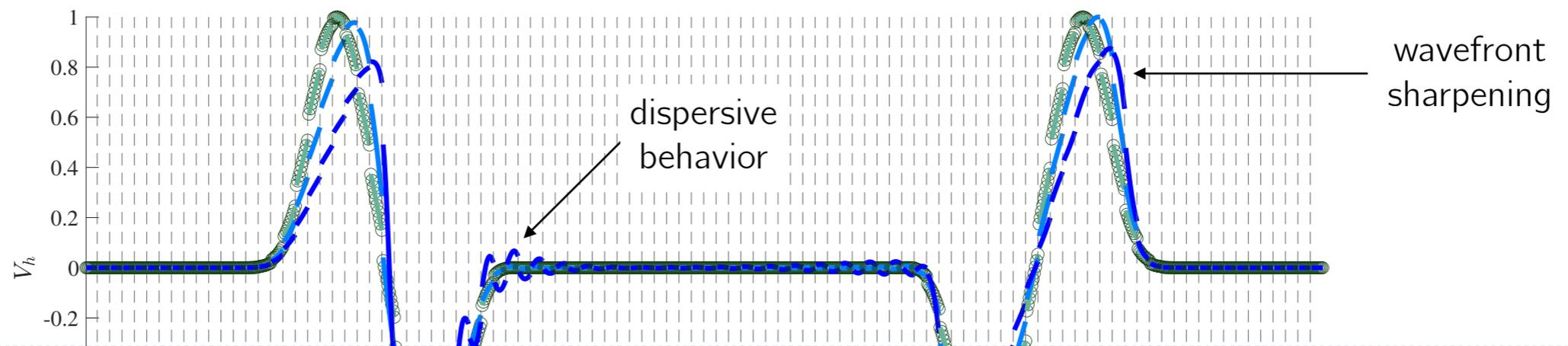
bulk mech. energy

interface energy

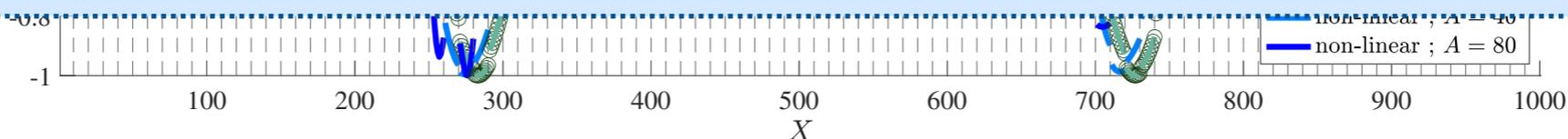
$$\mathcal{E}_h^m(t) = \frac{1}{2} \int_I \left\{ \rho_h(X) V_h(X, t)^2 + \frac{1}{E_h(X)} \Sigma_h(X, t)^2 \right\} dX$$

$$\mathcal{E}_h^i(t) = \sum_{X_n^I} \left\{ \frac{1}{2} M \langle\langle V_h(\cdot, t) \rangle\rangle_{X_n^I}^2 + K \int_0^{\mathcal{R}^{-1}(\langle\langle \Sigma_h(\cdot, t) \rangle\rangle_{X_n^I} / K)} \mathcal{R}(\zeta) d\zeta \right\}$$

► Numerical simulation: source with amplitude  $A$  and *fixed* central frequency

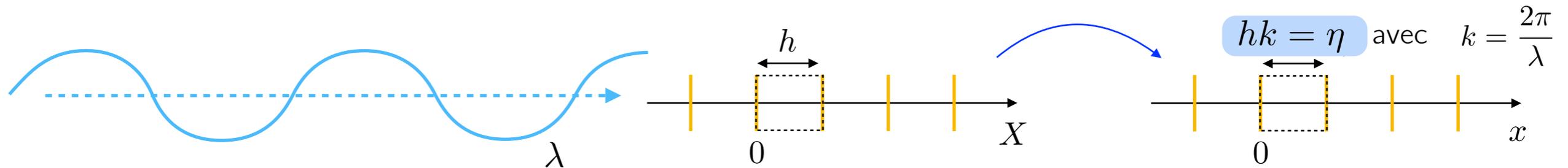


➔ 1<sup>st</sup>-order approximation in the case of *small* source amplitudes and at (*short*) finite times



# Two-scale analysis

- Non-dimensionalization of governing equations:  $(X, t) \longrightarrow (x, \tau)$



- wave equation:  $\alpha \left( \frac{x}{\eta} \right) \frac{\partial^2 u_\eta}{\partial \tau^2} (x, \tau) = \frac{\partial}{\partial x} \left( \beta \left( \frac{x}{\eta} \right) \frac{\partial u_\eta}{\partial x} (x, \tau) \right) + f(x, \tau)$

- interface conditions: 
$$\begin{cases} m \eta \left\langle \left\langle \frac{\partial^2 u_\eta}{\partial \tau^2} (\cdot, \tau) \right\rangle \right\rangle_{x_n} = \left[ \beta \frac{\partial u_\eta}{\partial x} (\cdot, \tau) \right]_{x_n} \\ \left\langle \left\langle \beta \frac{\partial u_\eta}{\partial x} (\cdot, \tau) \right\rangle \right\rangle_{x_n} = \frac{k}{h} \mathcal{R} \left( \frac{h}{\eta} \llbracket u_\eta(\cdot, \tau) \rrbracket_{x_n} \right) \end{cases}$$

uncover all contributions in  $\eta$

- Formal asymptotic expansion:  $u_\eta(x, \tau) = u_0(x, \tau) + \sum_{j \geq 1} \eta^j u_j(x, x/\eta, \tau)$  with fast variable  $y = x/\eta$

$\eta \ll 1$

**! Specificities of the setting considered:**

- smooth interface law: Taylor exp.  $\mathcal{R} \left( \frac{h}{\eta} \llbracket u_\eta \rrbracket_{y_n} \right) = \sum_{\ell \geq 0} \frac{(h\eta)^\ell}{\ell!} \left( \sum_{j \geq 2} \eta^{j-2} \llbracket u_j(x, \cdot, \tau) \rrbracket_{y_n} \right)^\ell \mathcal{R}^{(\ell)} \left( h \llbracket u_1(x, \cdot, \tau) \rrbracket_{y_n} \right)$
- 1D magic: direct integration possible and  $\left\langle \left\langle \frac{dg}{dy} \right\rangle \right\rangle = \int_0^1 \frac{dg}{dy} (y) dy = - \llbracket g \rrbracket$

# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

## ▸ Zeroth-order field $U_0$ :

- continuous in  $X$

- solution of:  $\rho_{\text{eff}} \frac{\partial^2 U_0}{\partial t^2}(X, t) = \frac{\partial \Sigma_0}{\partial X}(X, t) + F(X, t)$

effective mass density

$$\rho_{\text{eff}} = \left( \langle \rho \rangle + \frac{M}{h} \right)$$

local and **non-linear stress-strain relation**  $\Sigma_0 = \mathcal{G}_{\text{eff}}(\mathcal{E}_0)$

$$\left\langle \frac{1}{E} \right\rangle \Sigma_0(X, t) + \frac{1}{h} \mathcal{R}^{-1} \left( \frac{1}{K} \Sigma_0(X, t) \right) = \mathcal{E}_0(X, t)$$

where  $\mathcal{E}_0 = \partial U_0 / \partial X$

## ▸ Remarks:

- in the case of **linear interfaces**, i.e.  $\mathcal{R}(\zeta) = \zeta$

$$\mathcal{G}_{\text{eff}}(\mathcal{E}_0) = \mathcal{C}_{\text{eff}}^{\ell} \mathcal{E}_0 \quad \text{with} \quad \mathcal{C}_{\text{eff}}^{\ell} = \left( \left\langle \frac{1}{E} \right\rangle + \frac{1}{Kh} \right)^{-1}$$

- in the case of **perfect interfaces**, i.e.  $K \rightarrow +\infty$ ,  $M \rightarrow 0$

$$\rho_{\text{eff}} \sim \langle \rho \rangle \quad \text{and} \quad \mathcal{C}_{\text{eff}}^{\ell} \sim \langle 1/E \rangle^{-1}$$

# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

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where  $\mathcal{E}_0 = \partial U_0 / \partial X$

▶ Rewriting as a **non-linear first-order system**:

$$\frac{\partial}{\partial t} \Psi_0(X, t) + \frac{\partial}{\partial X} \left( \mathbb{G}_{\text{eff}}(\Psi_0(X, t)) \right) = \mathbb{F}(X, t) \quad \text{with} \quad \Psi_0 = (\mathcal{E}_0, V_0)^\top$$

- strictly hyperbolic system (owing to technical assumptions on  $\mathcal{R}$ )

- characteristic speeds (possibly strain dependent)  $\nu_{\pm}(\mathcal{E}_0) = \pm \sqrt{\frac{1}{\rho_{\text{eff}}} \frac{\partial \Sigma_0}{\partial \mathcal{E}_0}}$

# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

▶ Zeroth-order field  $U_0$ :

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effective mass density

$$\rho_{\text{eff}} = \left( \langle \rho \rangle + \frac{M}{h} \right)$$

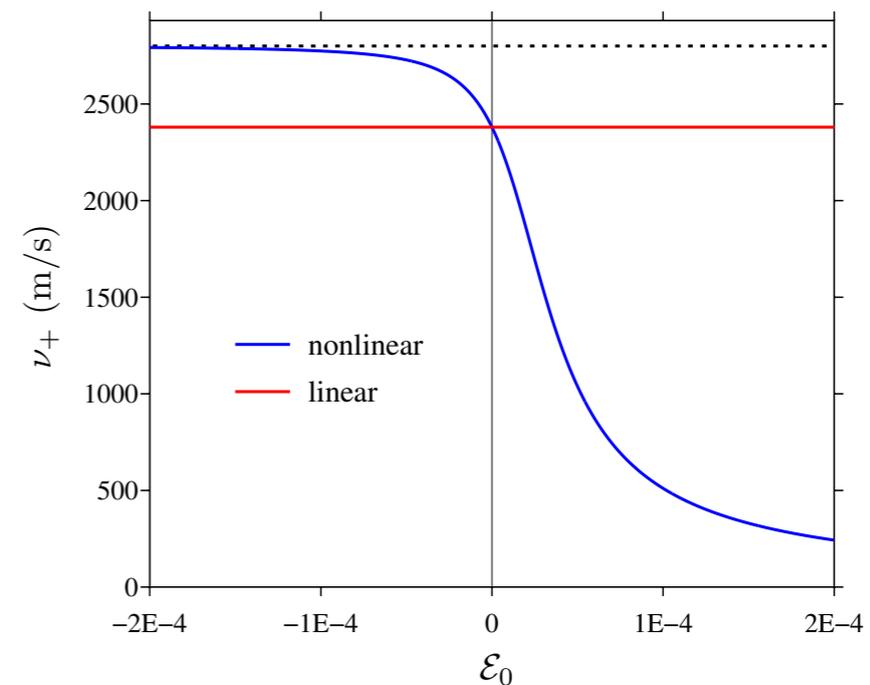
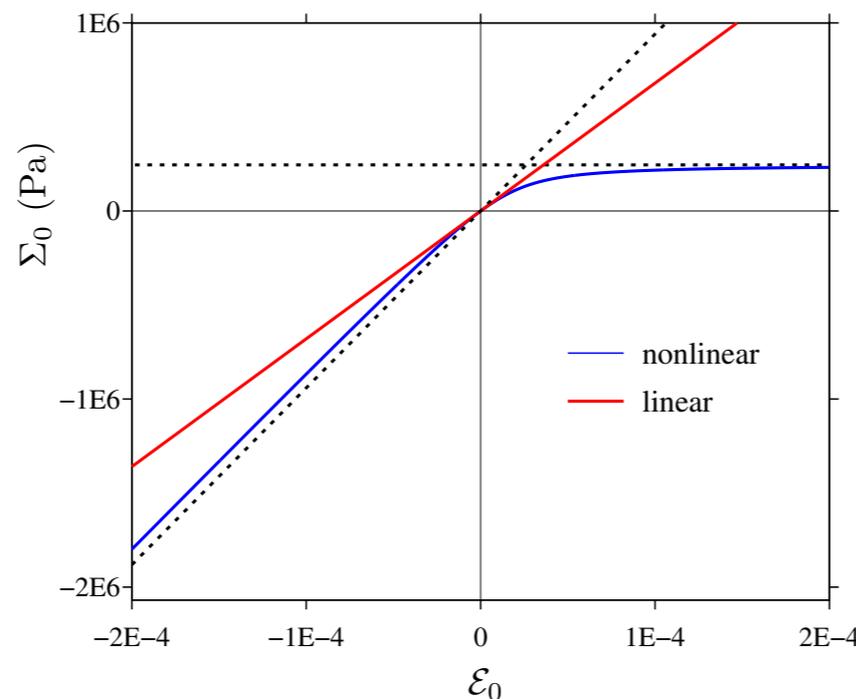
local and **non-linear stress-strain relation**  $\Sigma_0 = \mathcal{G}_{\text{eff}}(\mathcal{E}_0)$

$$\left\langle \frac{1}{E} \right\rangle \Sigma_0(X, t) + \frac{1}{h} \mathcal{R}^{-1} \left( \frac{1}{K} \Sigma_0(X, t) \right) = \mathcal{E}_0(X, t)$$

▶ Examples:

$$\mathcal{R}(\zeta) = \zeta$$

$$\mathcal{R}(\zeta) = \frac{\zeta}{1 + \zeta/d}$$



# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

▸ Zeroth-order field  $U_0$  :

$$\frac{\partial}{\partial t} \Psi_0(X, t) + \frac{\partial}{\partial X} \left( \mathbb{G}_{\text{eff}}(\Psi_0(X, t)) \right) = \mathbb{F}(X, t) \quad \text{with} \quad \Psi_0 = (\mathcal{E}_0, V_0)^\top$$

▸ Energy analysis:

- recall  $\frac{d}{dt} \mathcal{E}_h = 0$  in microstructured medium

- here:  $\mathcal{E}_0(t) = \int_I \left\{ \frac{1}{2} \rho_{\text{eff}} V_0^2 + \mathcal{J}_{\text{eff}} \right\} dX$  with  $\mathcal{J}_{\text{eff}}(\mathcal{E}_0) = \int_0^{\mathcal{E}_0} \mathcal{G}_{\text{eff}}(\tilde{\mathcal{E}}_0) d\tilde{\mathcal{E}}_0$

# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

## ▸ Zeroth-order field $U_0$ :

$$* \quad \frac{\partial}{\partial t} \Psi_0(X, t) + \frac{\partial}{\partial X} \left( \mathbb{G}_{\text{eff}}(\Psi_0(X, t)) \right) = \mathbb{F}(X, t) \quad \text{with} \quad \Psi_0 = (\mathcal{E}_0, V_0)^\top$$

## ▸ Energy analysis:

- recall  $\frac{d}{dt} \mathcal{E}_h = 0$  in microstructured medium

- here:  $\mathcal{E}_0 = \mathcal{E}_0^m + \mathcal{E}_0^i$  such that, for *sufficiently smooth fields*, it holds  $\frac{d}{dt} \mathcal{E}_0 = 0$  without source

$$\text{bulk energy } \mathcal{E}_0^m(t) = \frac{1}{2} \int_I \left\{ \langle \rho \rangle V_0^2 + \left\langle \frac{1}{E} \right\rangle \Sigma_0^2 \right\} dX$$

$$\text{interface energy } \mathcal{E}_0^i(t) = \frac{1}{h} \int_I \left\{ \frac{1}{2} M V_0^2 + K \int_0^{\mathcal{R}^{-1}(\Sigma_0/K)} \mathcal{R}(\zeta) d\zeta \right\} dX$$



\* formally analogous to so-called *p-system* in gas dynamics ➔

 Lax, Dafermos, ...

**formation of shocks**

in finite time  $t^*$

$$\frac{d}{dt} \mathcal{E}_0 \leq 0 \quad \text{for} \quad t \geq t^*$$

↪ for linearized stress-strain relation  $\Sigma_0 \underset{\mathcal{E}_0 \rightarrow 0}{\sim} \mathcal{C}_{\text{eff}}^\ell \mathcal{E}_0 (1 - \gamma \mathcal{E}_0) + o(\mathcal{E}_0^2)$

time-harmonic forcing at  $\omega_c$  and amplitude  $\mathcal{E}_{\text{max}}$

⇒

$$t^* \propto \frac{1}{\mathcal{E}_{\text{max}} \gamma \omega_c}$$

# Microstructured field approximation

➔ 1<sup>st</sup>-order approximation:  $U_h(X, t) = U_0(X, t) + h U_1(X, t) + o(h)$

► **First-order field  $U_1$  (corrector):**

- decomposed as  $U_1(X, t) = \bar{U}_1(X, t) + \mathcal{P}(y, \mathcal{E}_0(X, t)) \mathcal{E}_0(X, t)$

↙
↘

mean field
cell-function

► **Mean field:** solution of **linear** and **heterogeneous problem**

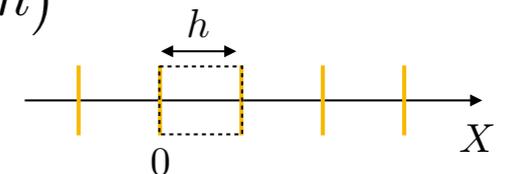
\*  $\rho_{\text{eff}} \frac{\partial^2 \bar{U}_1}{\partial t^2}(X, t) = \frac{\partial \bar{\Sigma}_1}{\partial X}(X, t) + \mathcal{S}(U_0(X, t))$  with  $\bar{\Sigma}_1(X, t) = \mathcal{G}'_{\text{eff}}(\mathcal{E}_0(X, t)) \frac{\partial \bar{U}_1}{\partial X}(X, t)$

↘
↙

source term and stiffness (may) depend non-linearly on  $U_0$

► **Cell-function:**  $\mathcal{P}(y, \mathcal{E}_0(X, t))$  where  $y = (X - nh)/h$  for  $X \in (nh, (n+1)h)$

(possibly) non-linear fct of  $U_0$



► **Remarks:**

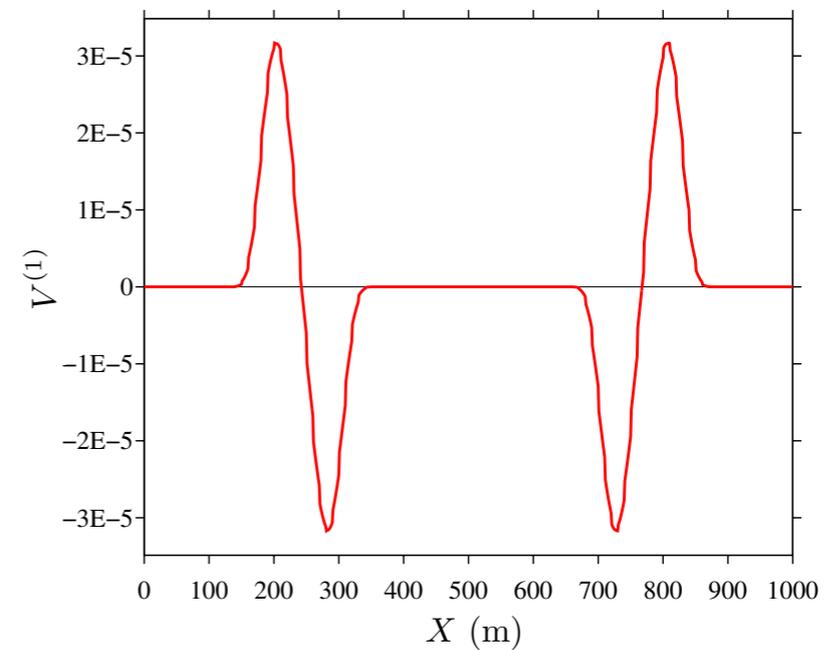
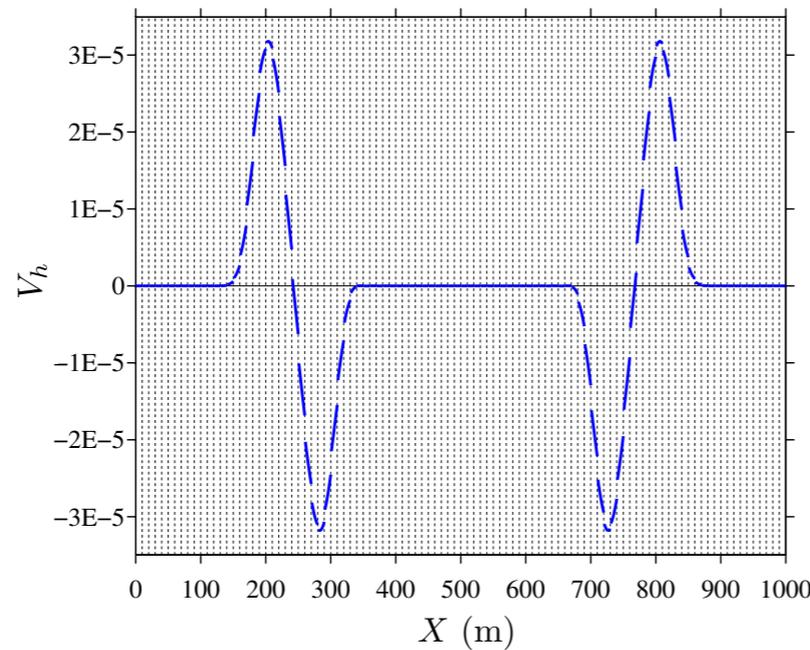
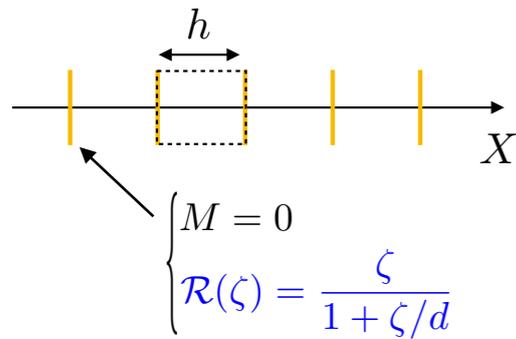
- problem \* is well-posed (owing to technical assumptions on  $\mathcal{R}$ )
- in the case of **linear interfaces**:  $\mathcal{G}'_{\text{eff}}(\mathcal{E}_0(X, t)) = \mathcal{C}_{\text{eff}}^l$  ;  $\mathcal{S}(U_0(X, t)) = 0$  ;  $\mathcal{P} \equiv \mathcal{P}(y)$
- rewriting of \* as a **linear** and **strictly hyperbolic system** with charac. speeds  $\nu_{\pm}(\mathcal{E}_0)$

# Numerical results:



**Objectives:** comparisons of full-field simulations (velocity-based)  $V_h$  with homogenized solutions, i.e.  $V_0$  and  $V^{(1)}(X, t) = V_0(X, t) + hV_1(X, t)$

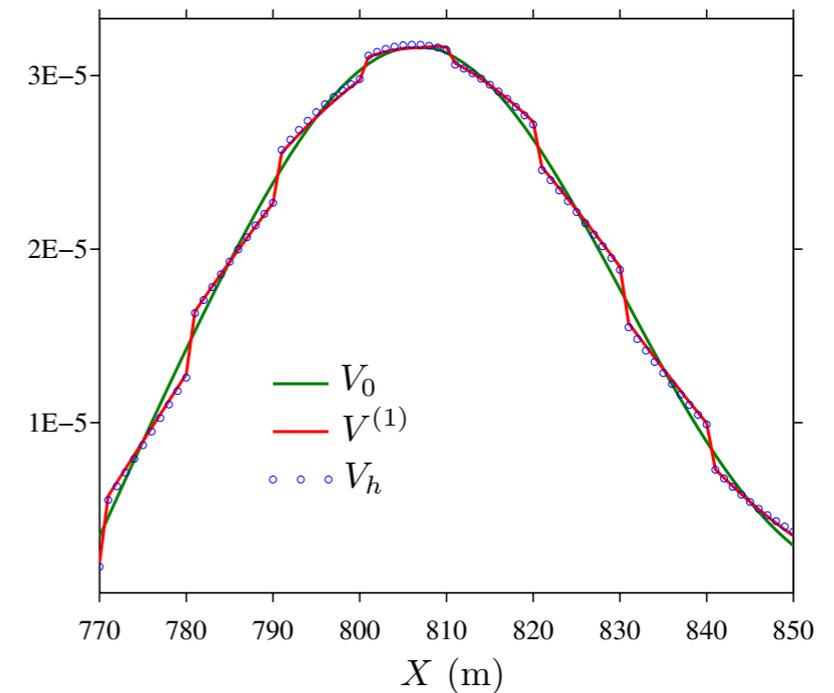
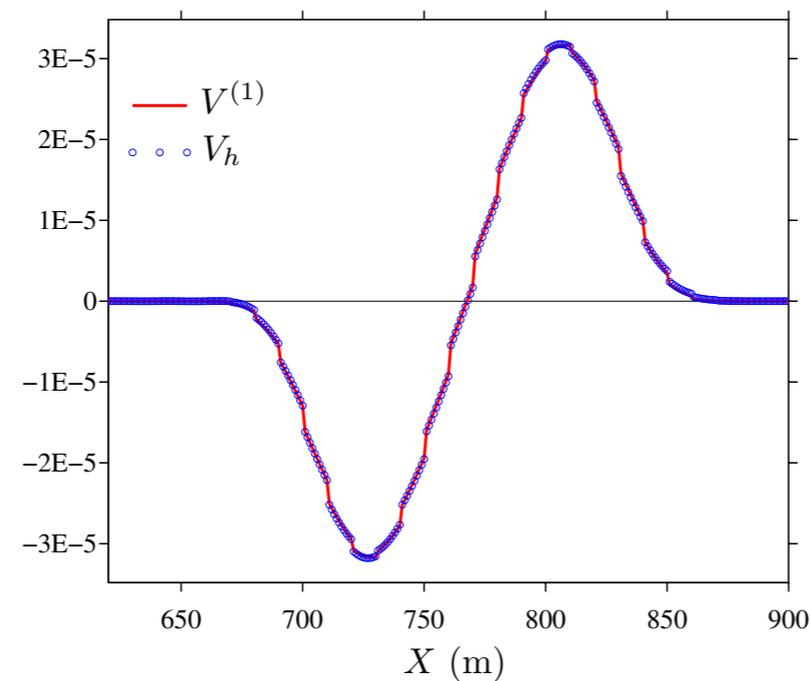
► **Medium:**



► **Excitation:**

$$\begin{cases} f_c = 10 \text{ Hz} \\ A = 0.1 \end{cases}$$

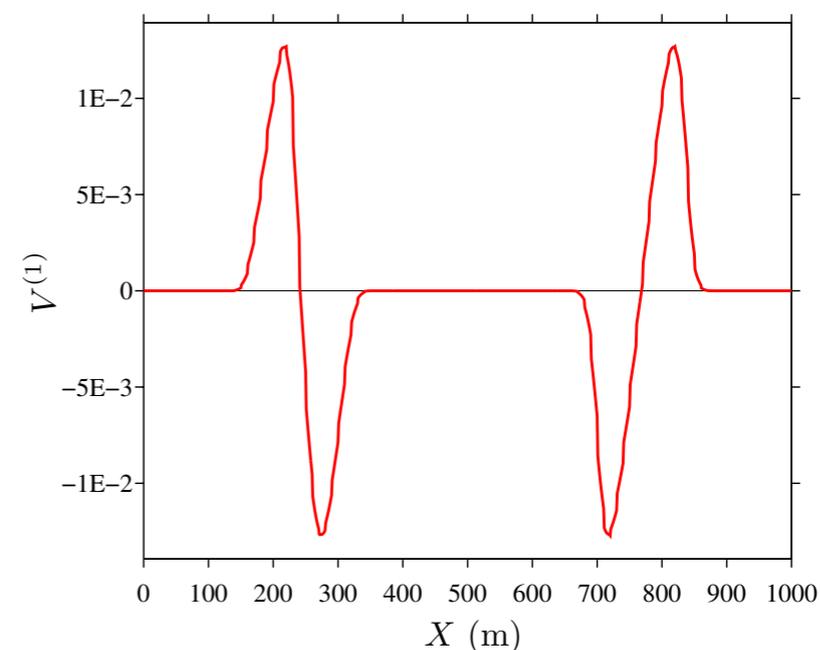
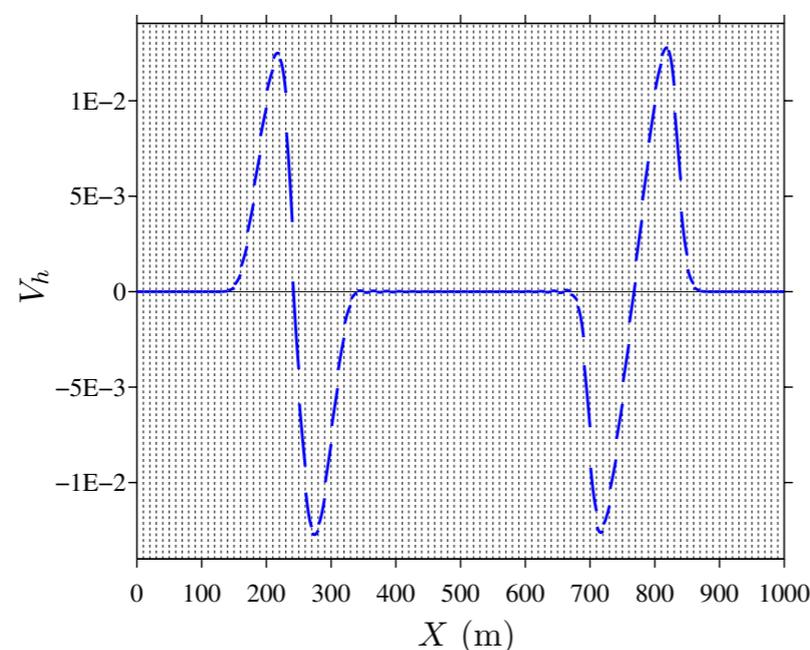
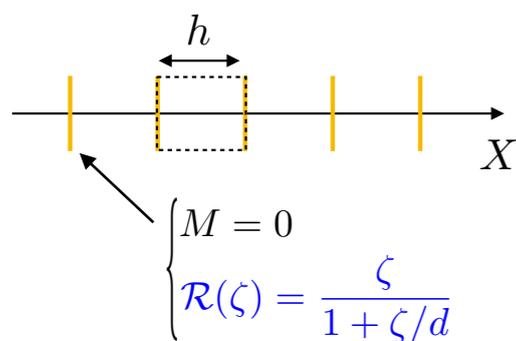
$$\eta = 0.26$$



# Numerical results: *agreements*

➔ Objectives: comparisons of full-field simulations (velocity-based)  $V_h$  with homogenized solutions, i.e.  $V_0$  and  $V^{(1)}(X,t) = V_0(X,t) + hV_1(X,t)$

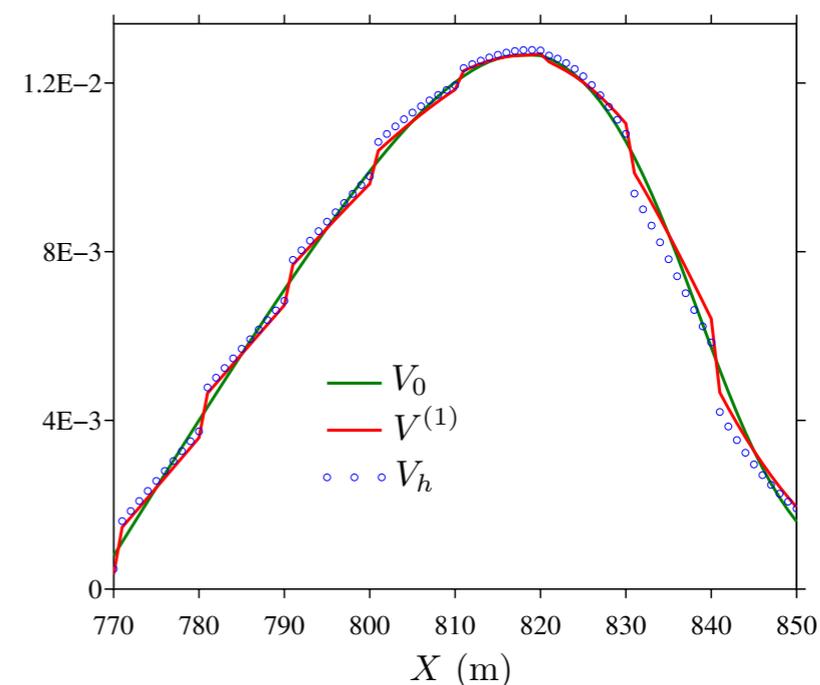
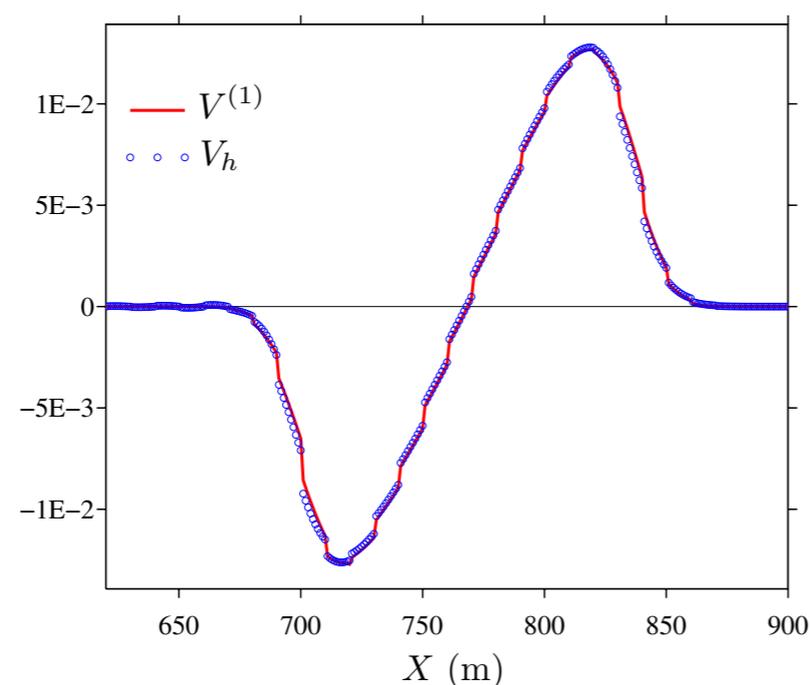
► Medium:



► Excitation:

$$\begin{cases} f_c = 10 \text{ Hz} \\ A = 40 \end{cases}$$

$$\eta = 0.26$$

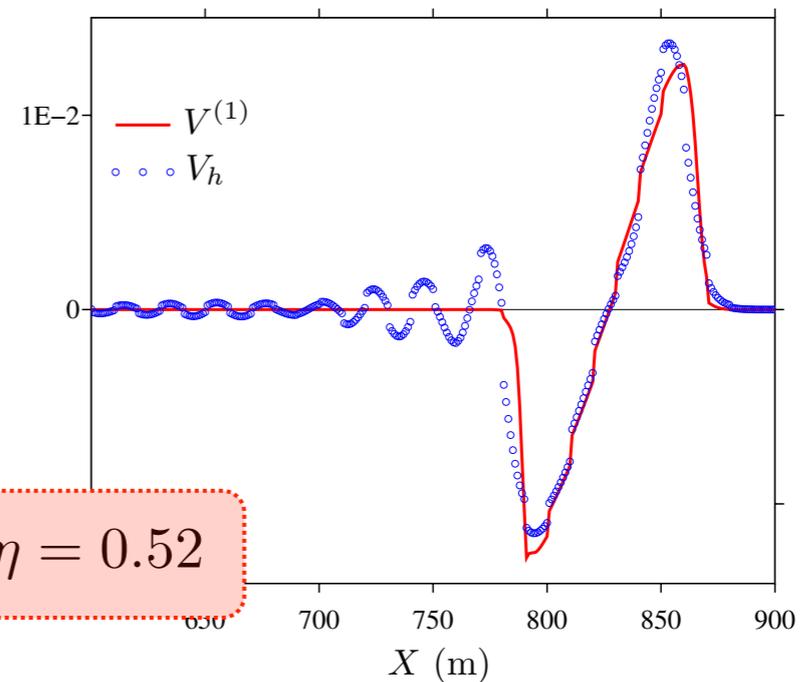
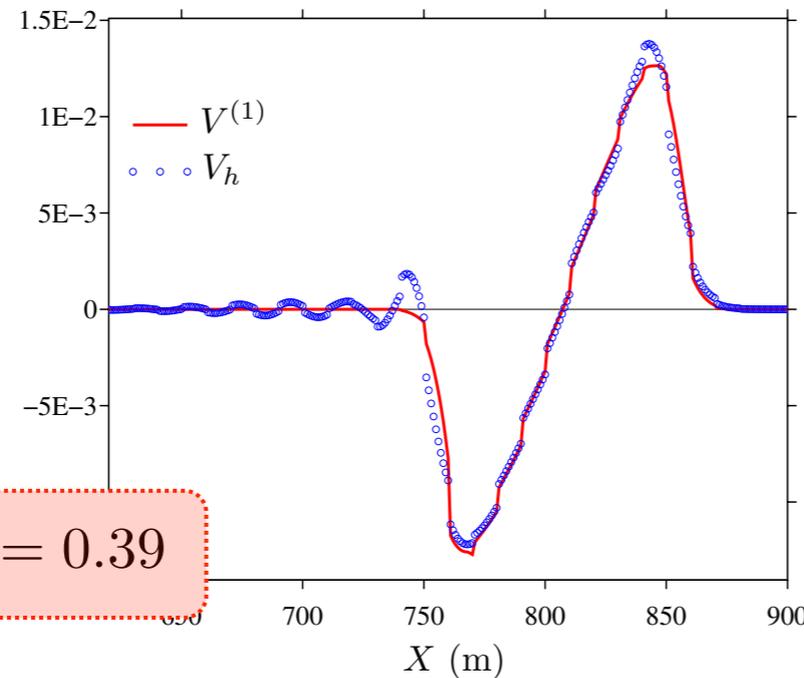


# Numerical results: ... and limitations

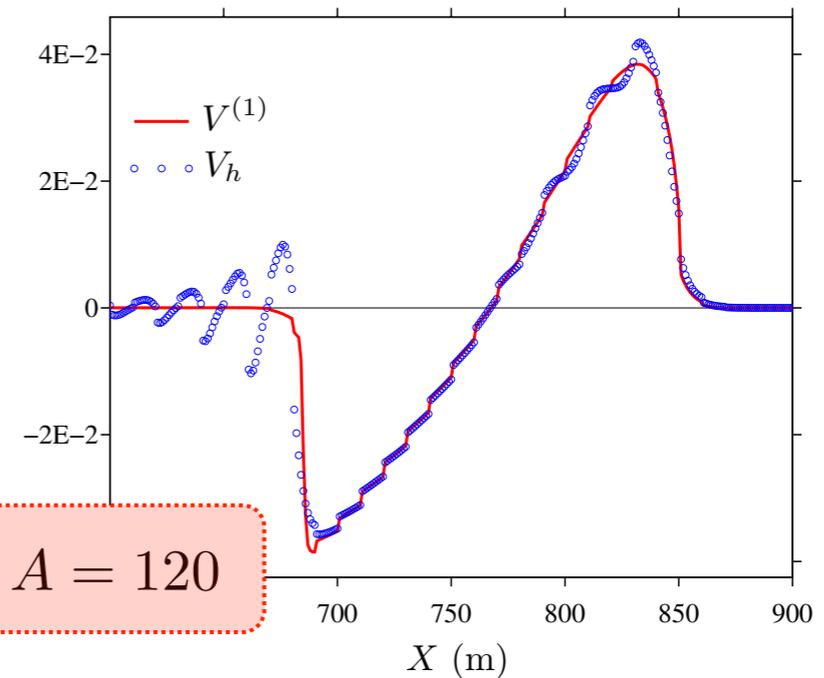
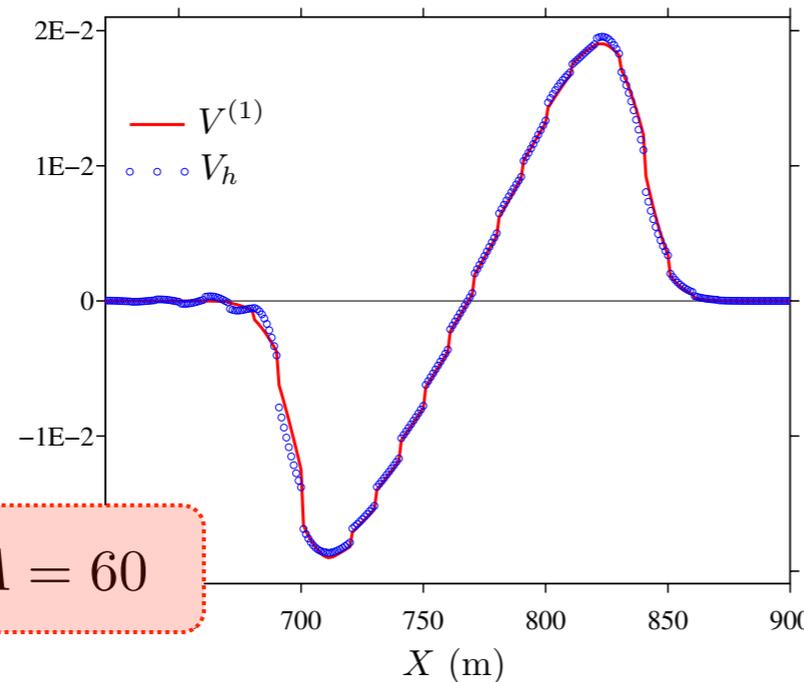


Objectives: comparisons of full-field simulations (velocity-based)  $V_h$  with homogenized solutions, i.e.  $V_0$  and  $V^{(1)}(X,t) = V_0(X,t) + hV_1(X,t)$

► Excitation:  $A = 40$



► Excitation:  $\eta = 0.26$

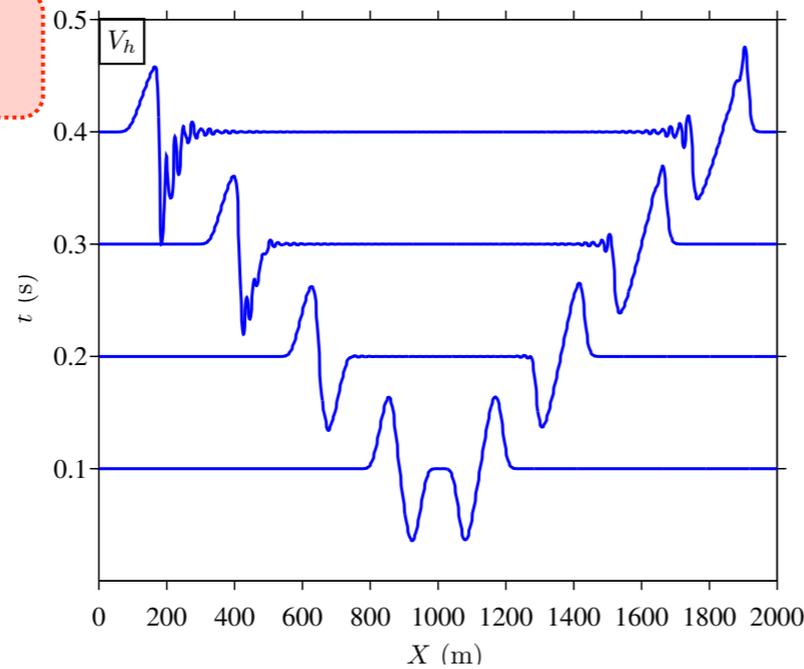


# Numerical results: formation of shocks

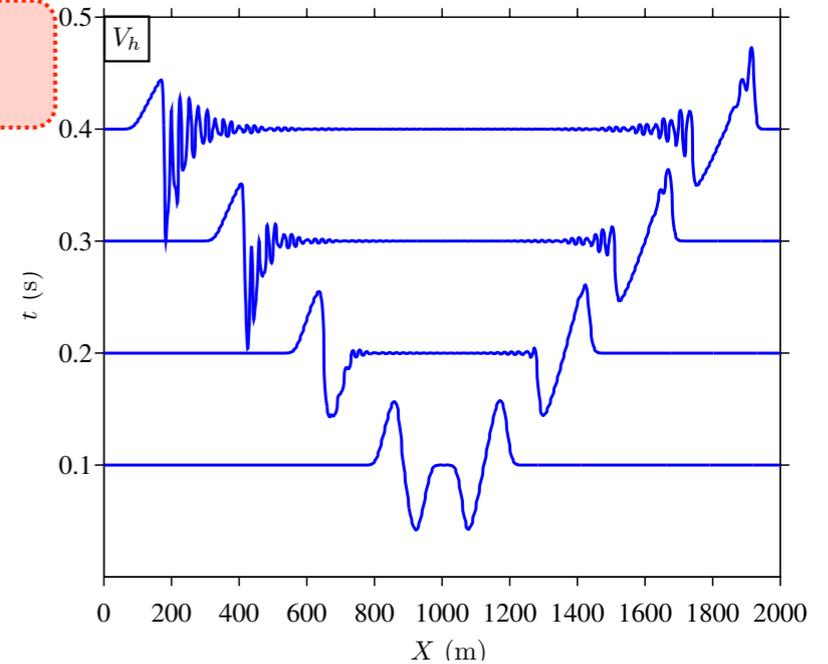


**Objectives:** comparisons of full-field simulations (velocity-based)  $V_h$  with homogenized solutions, i.e.  $V_0$  and  $V^{(1)}(X, t) = V_0(X, t) + hV_1(X, t)$

$A = 40$



$A = 60$

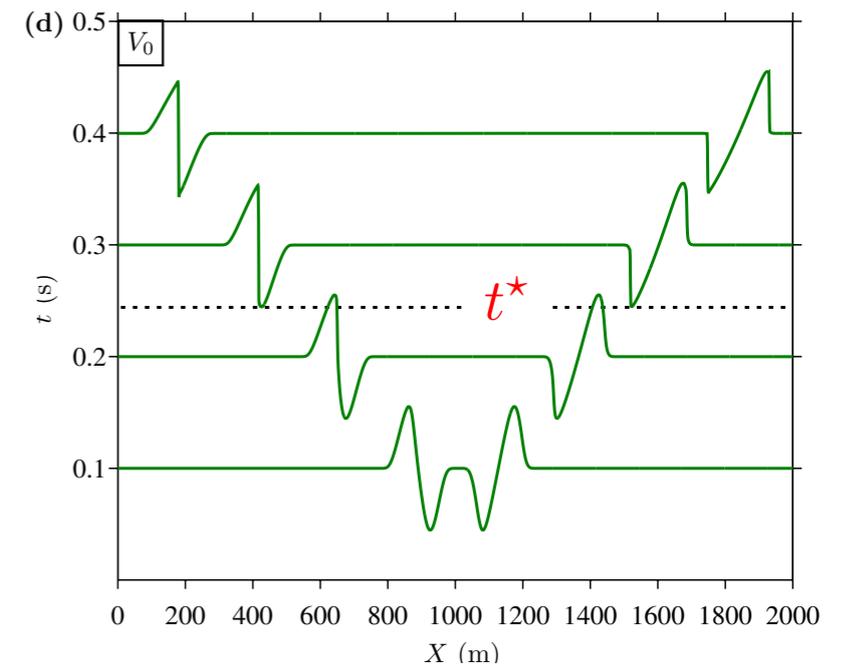
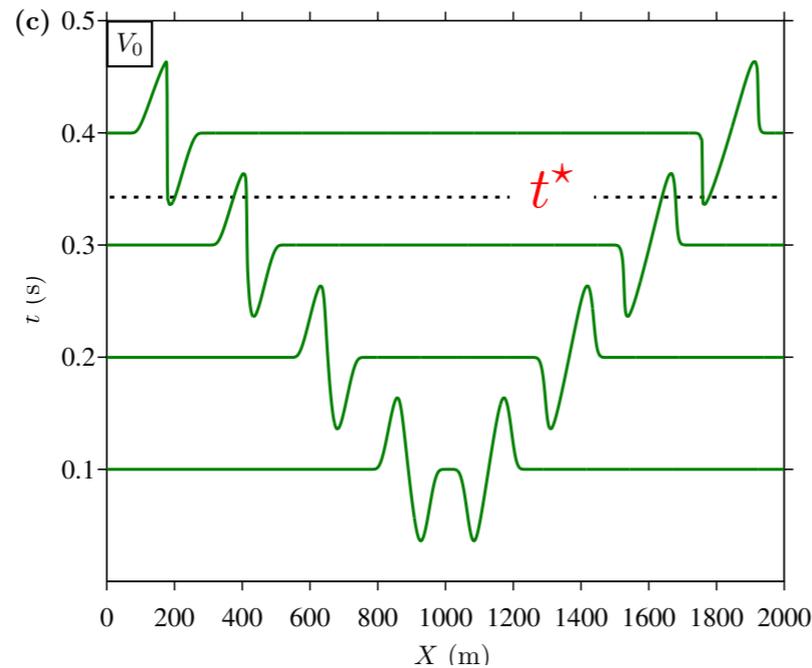


► What happens if shocks occur?

$t$



(as expected with effective model with estimated time  $t^*$ )

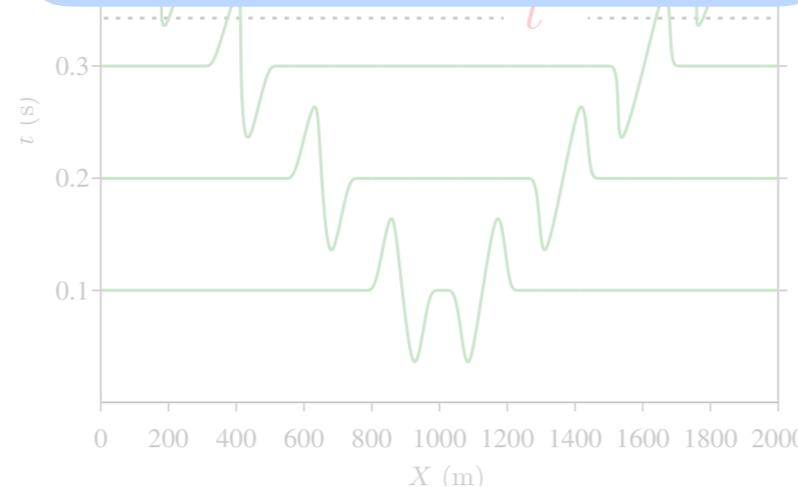
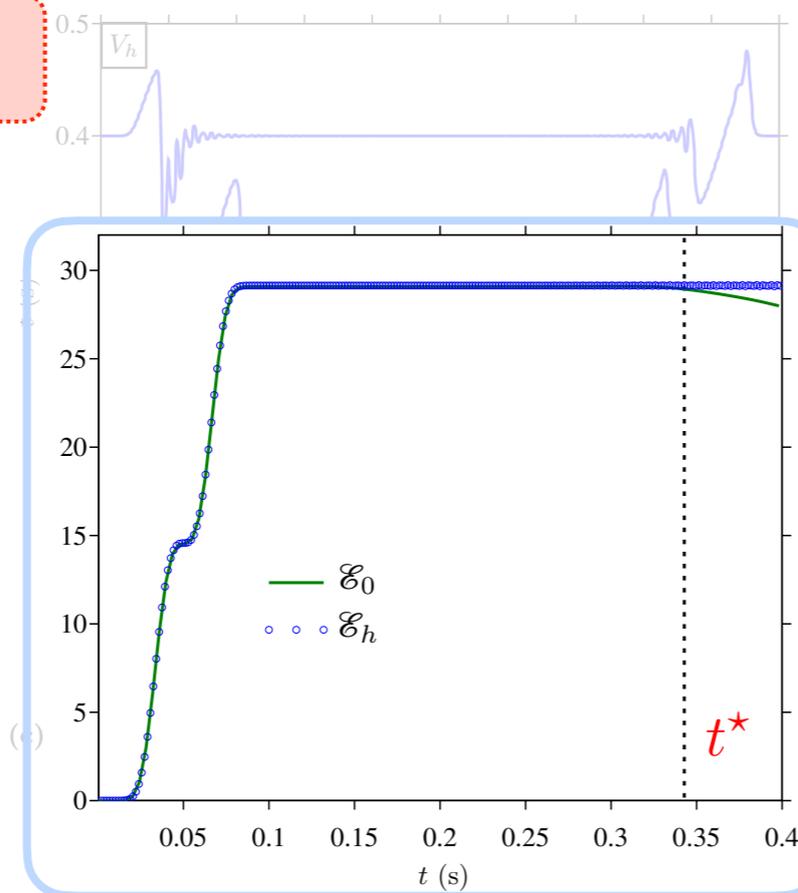


# Numerical results: formation of shocks

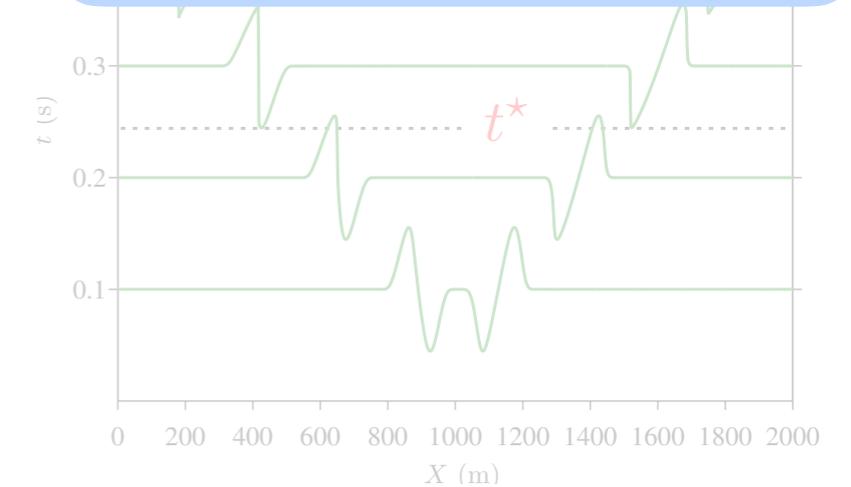
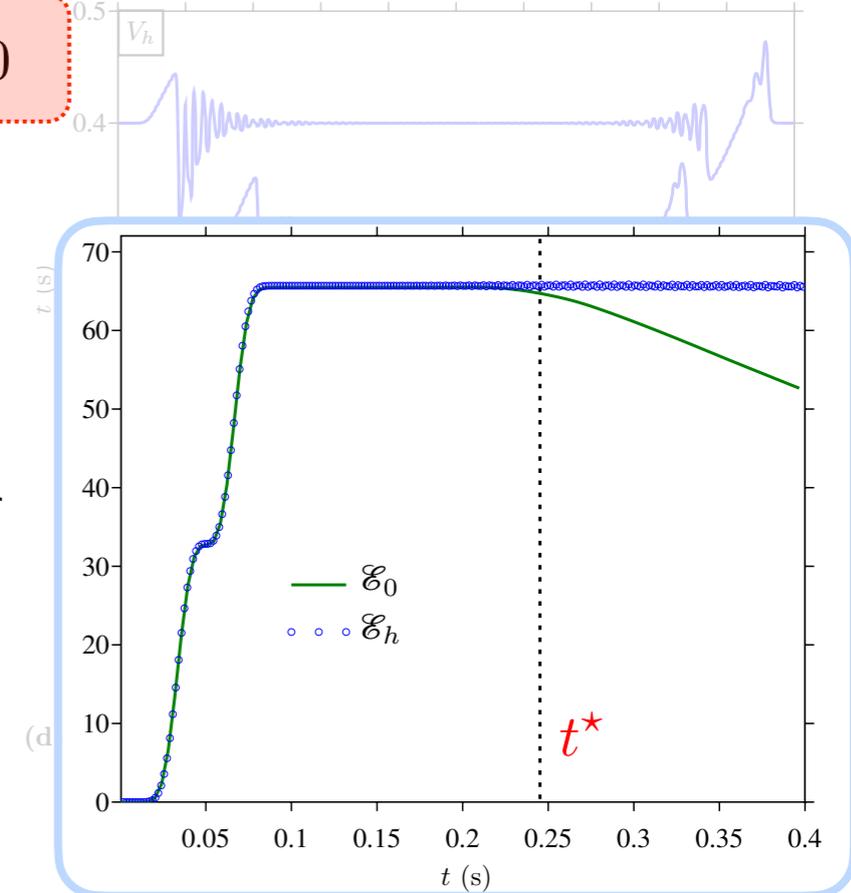


**Objectives:** comparisons of full-field simulations (velocity-based)  $V_h$  with homogenized solutions, i.e.  $V_0$  and  $V^{(1)}(X, t) = V_0(X, t) + hV_1(X, t)$

$A = 40$



$A = 60$



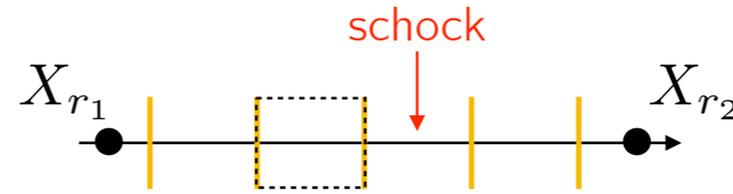
in terms of energies

► What happens if shocks occur?

(as expected with effective model with estimated time  $t^*$ )

# Numerical results: formation of shocks

- What about the accuracy of spectra?

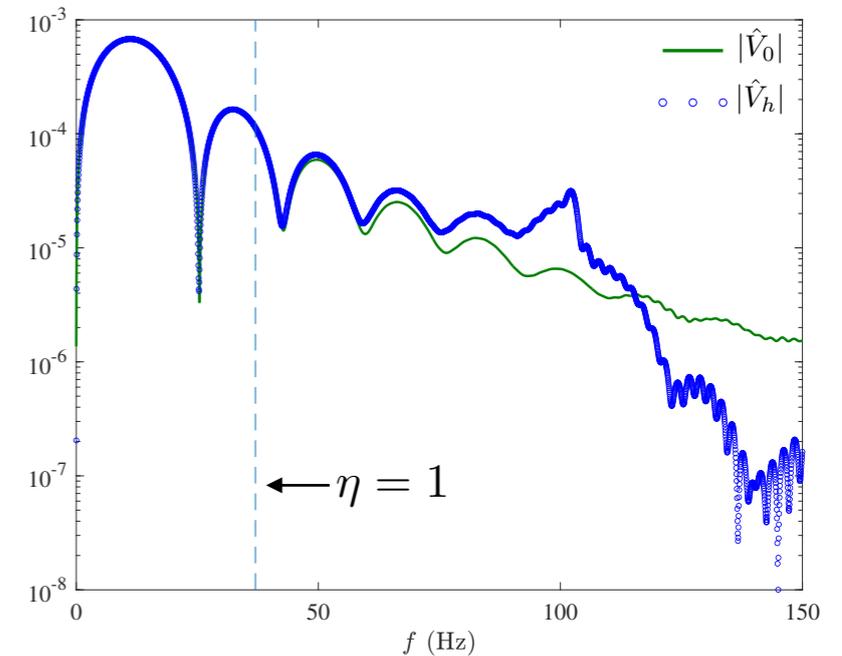
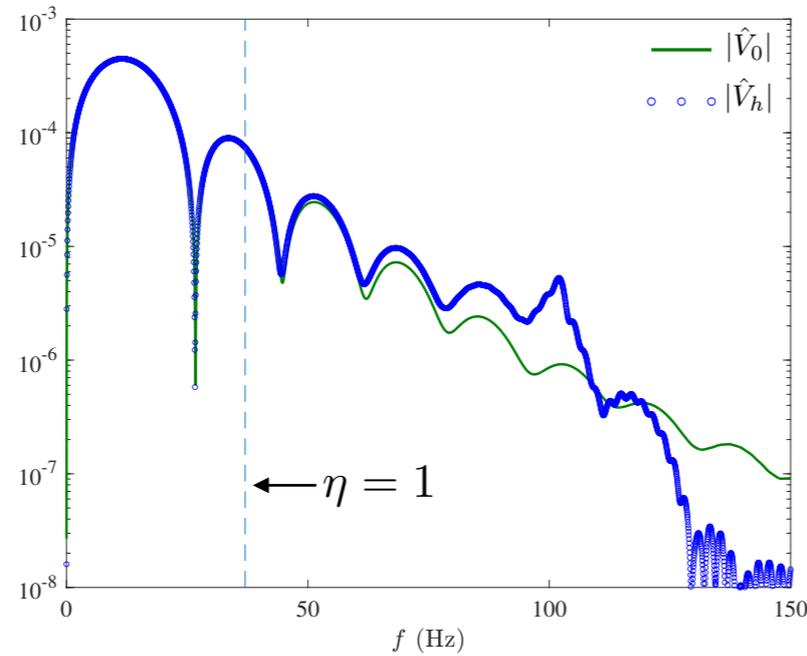


$A = 40$

$A = 60$

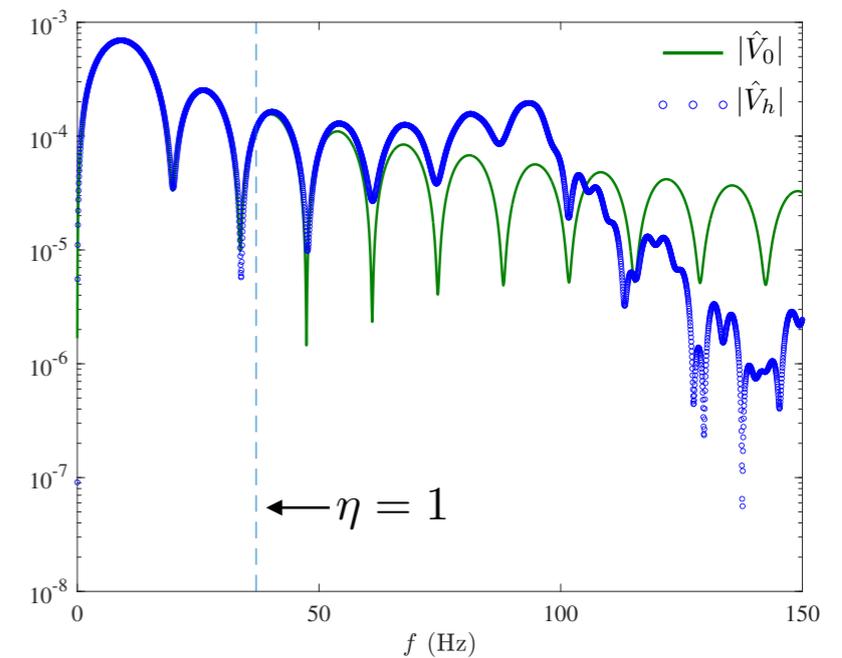
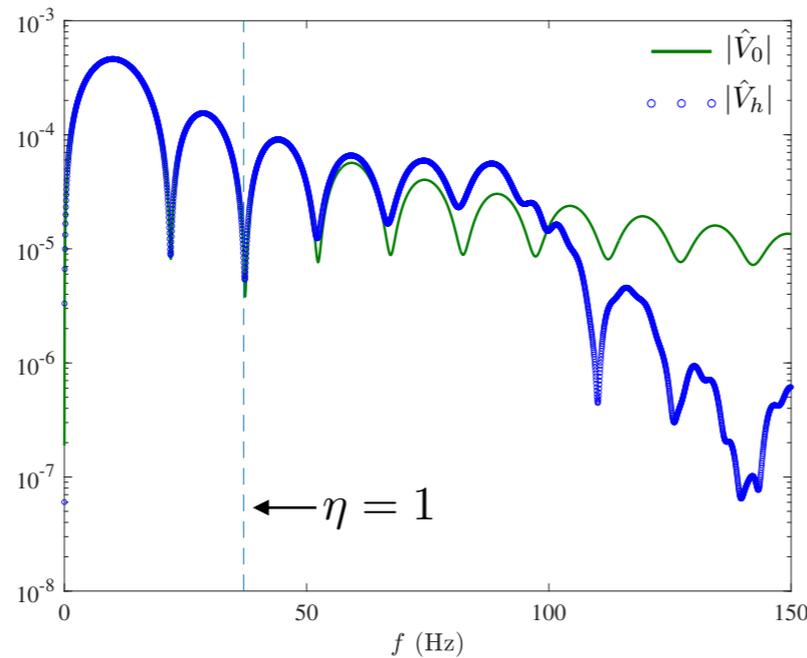
spectra  
before the shock

$X_{r_1}$



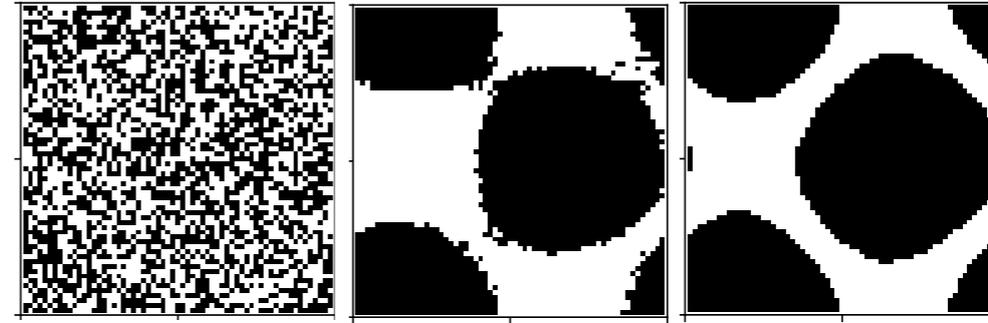
spectra  
after the shock

$X_{r_2}$

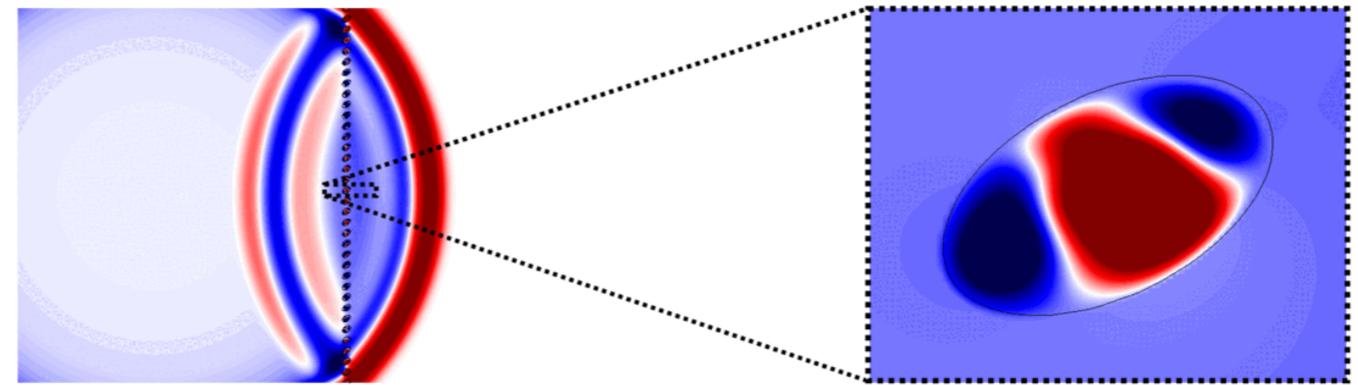


# Conclusion

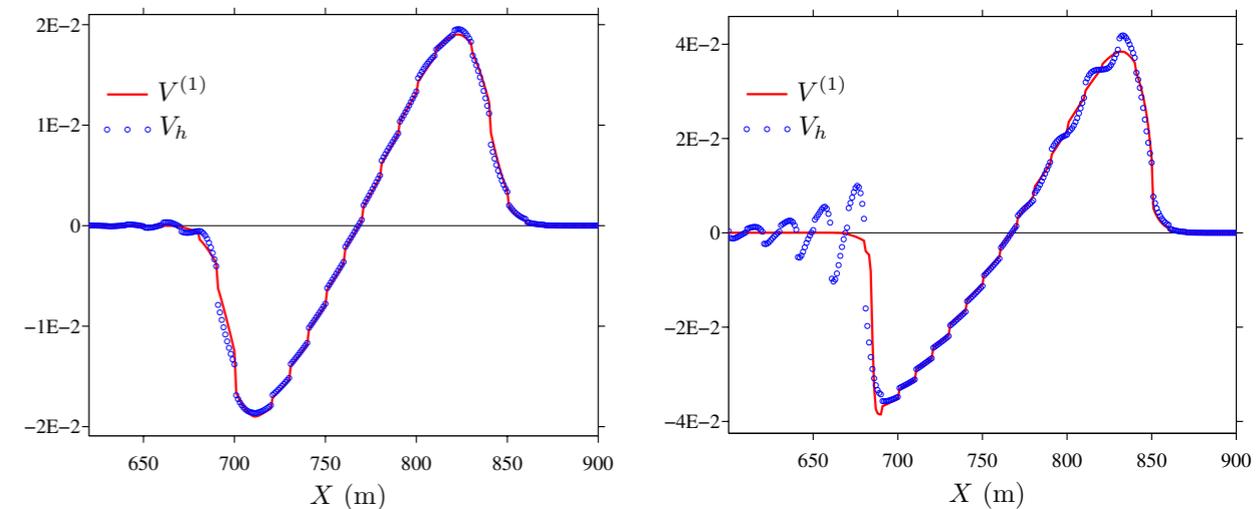
1. Dispersion, high-order homogenization and design of microstructures  
(harmonic regime)



2. Homogenization with internal resonances  
(time-domain)



3. Dynamical homogenization with non-linearities  
(time-domain)



- Perspectives:**
- metamaterial design (internal resonances, micro-architectures, transformation, ...)
  - high-frequency homogenization
  - time-modulated materials
  - topological optimization in resonant case



# Sur Quelques Travaux en Homogénéisation avec Inertie (et réciproquement)

Cédric Bellis<sup>1</sup>

jointly with Rémi Cornaggia<sup>2</sup>, Bruno Lombard<sup>1</sup>

Marie Touboul<sup>3</sup>, Raphaël Assier<sup>4</sup>

*Thank you for your attention*