Asymptotic homogenization and optimization of strain-gradient architected materials

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Université de Lorraine, Nancy, France

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"Architected materials designed with higher-order homogenization"

- Scientific objectives:
  - Derive a predictive homogenization scheme capturing strain-gradient elasto-static effective behavior
  - Generate new microstructures from topology optimization
  - Validate the non-standard behavior from experiments

- Partners:

Long range strain-gradient effects in elasto-statics?

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  - Always first-gradient equivalent medium at leading order
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Asymptotic expansion + infinite contrast?

Contents

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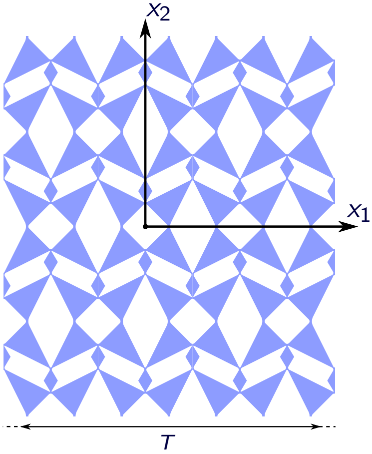

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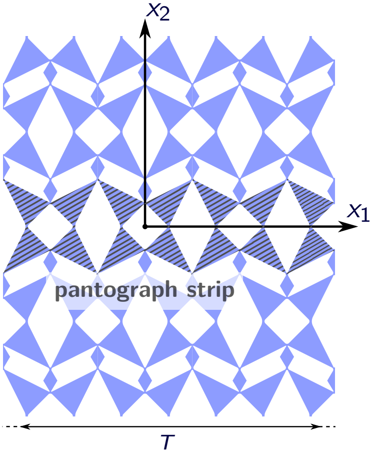

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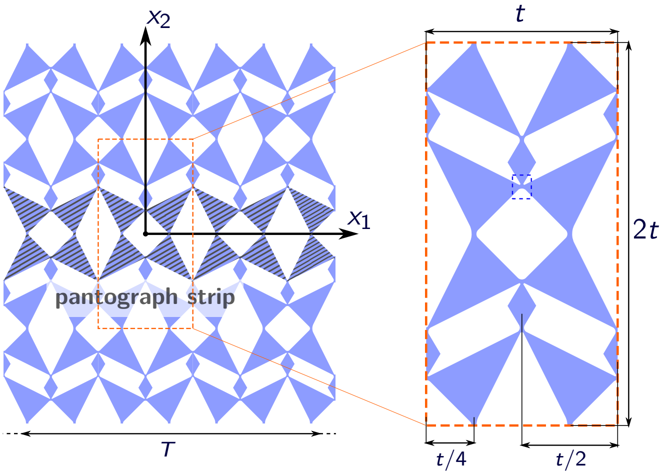

Scale ratio $\frac{t}{T} = \eta \to 0$

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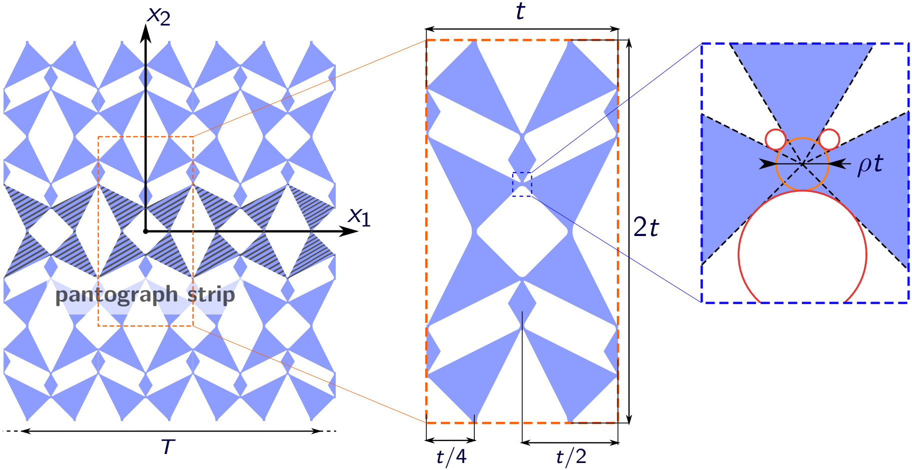

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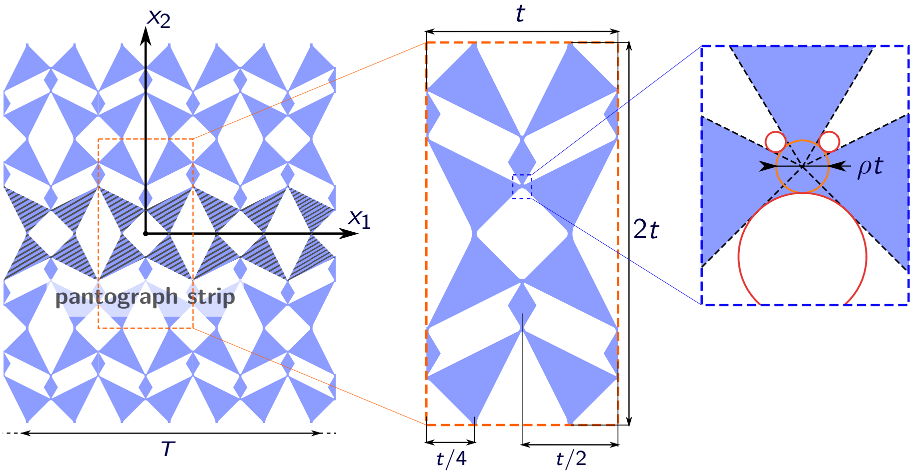
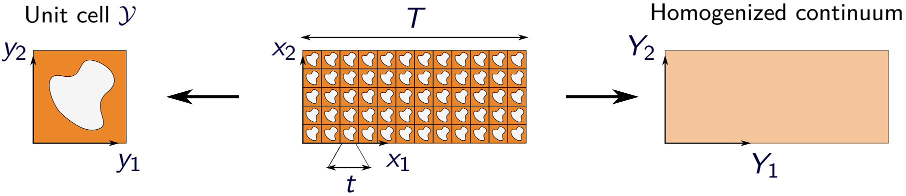


Scale ratio $\frac{t}{T} = \eta \to 0$ Junction thinness $\rho \to 0$

The two scale asymptotic expansion



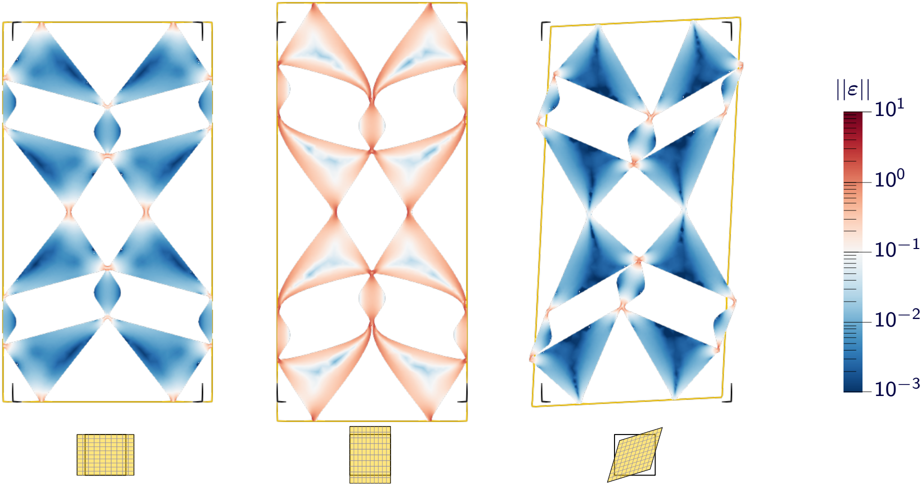
$y_i = \frac{x_i}{t}$ $\quad t \ll T \quad$ $Y_i = \frac{x_i}{T}$

- Scale ratio: $\eta = \frac{t}{T} \ll 1$
- Expansion: $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) = T \sum_{p=0}^{\infty} \eta^p \underline{\boldsymbol{u}}^p(\underline{\boldsymbol{Y}}, \underline{\boldsymbol{y}})$
- Differential operators: $\underline{\boldsymbol{\nabla}}_x = \frac{1}{T}\left(\underline{\boldsymbol{\nabla}}_Y + \frac{1}{\eta}\underline{\boldsymbol{\nabla}}_y\right)$

$\Rightarrow$ Series of **unit-cell problems** loaded by: $\underline{\boldsymbol{U}}(\underline{\boldsymbol{Y}})$, $\boldsymbol{E}(\underline{\boldsymbol{Y}})$, $\boldsymbol{K}(\underline{\boldsymbol{Y}})$...

First-order unit-cell problems

$$\boldsymbol{E} = \nabla\, \underline{\boldsymbol{U}}, \qquad \underline{\boldsymbol{u}}^{\mathrm{E}}(\underline{\boldsymbol{y}}) = \boldsymbol{E}\cdot\underline{\boldsymbol{y}} + \eta \boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E} \qquad \varepsilon^{\mathrm{E}}(\underline{\boldsymbol{y}}) = \boldsymbol{a}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E}$$



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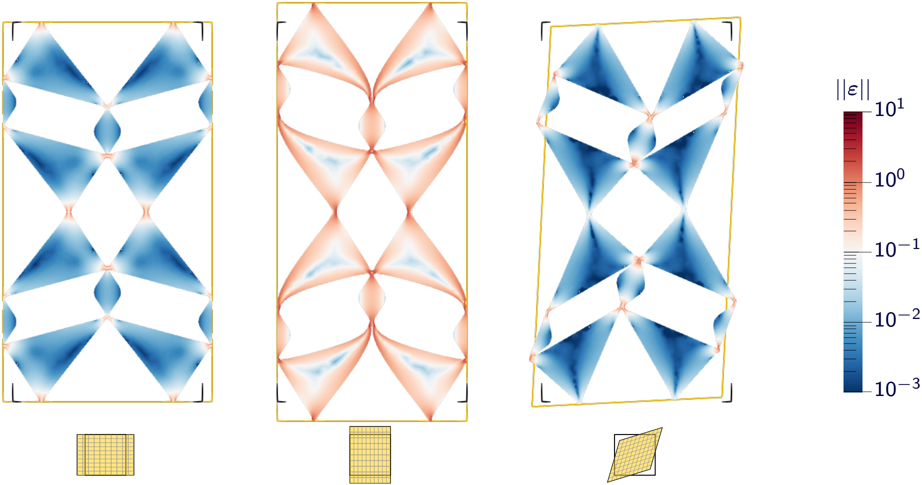

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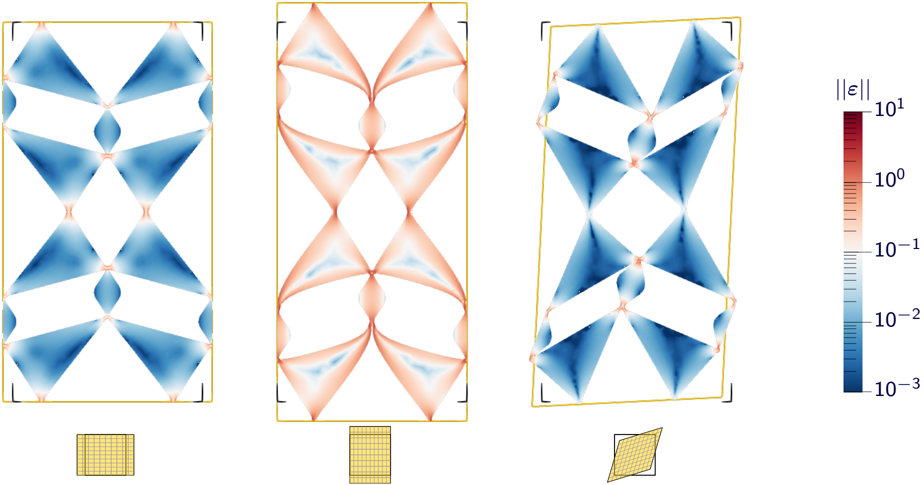

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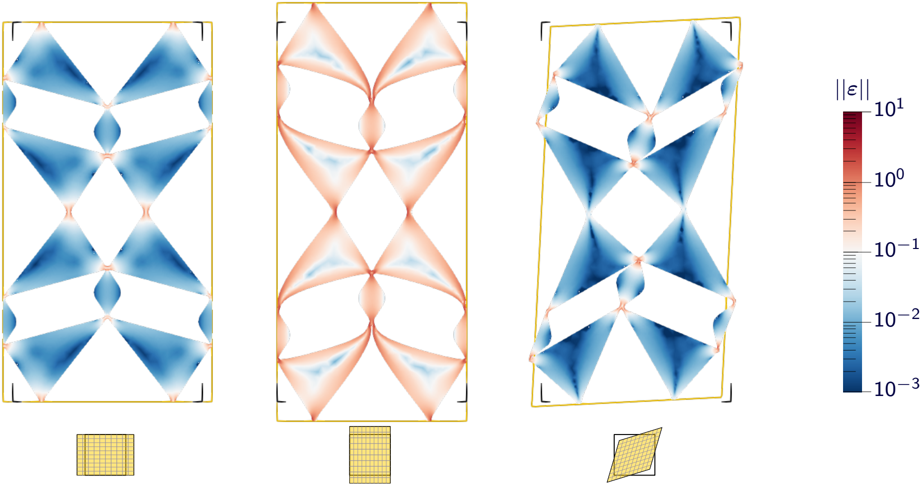
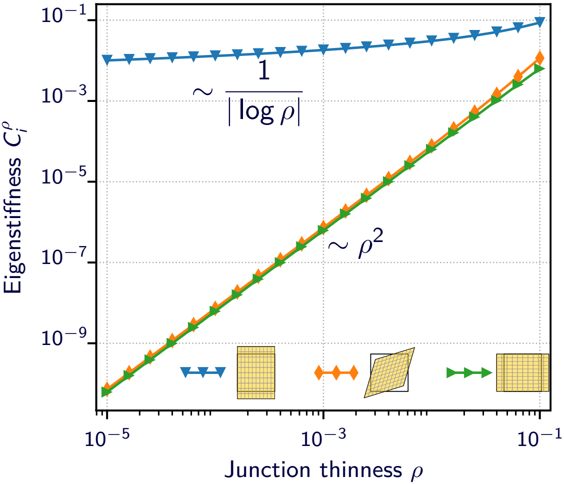


First-gradient homogenized energy

$$W^\rho = \frac{1}{2}\boldsymbol{E} : \boldsymbol{C}^\rho : \boldsymbol{E}$$

$$\boldsymbol{C}^\rho = \left\langle \left(\boldsymbol{a}^1\right)^{\mathsf{T}} : c(\underline{\boldsymbol{y}}) : \boldsymbol{a}^1 \right\rangle$$

$$\boldsymbol{C}^\rho = \sum_{i=1}^{3} C_i^\rho \boldsymbol{E}_i \otimes \boldsymbol{E}_i$$



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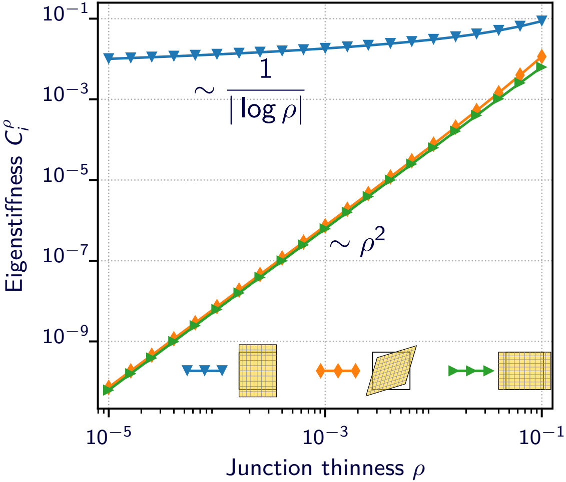

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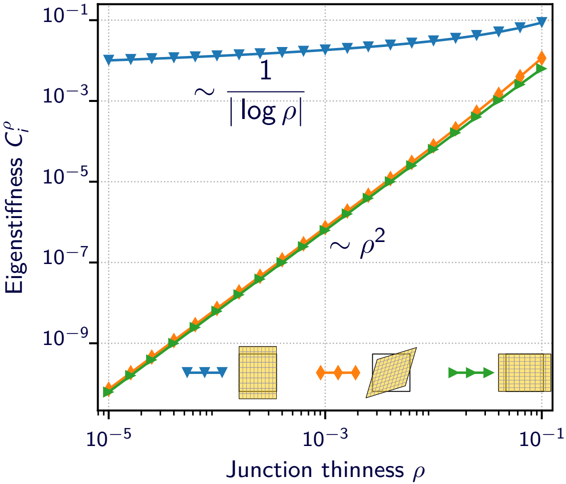


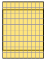
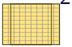
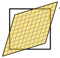
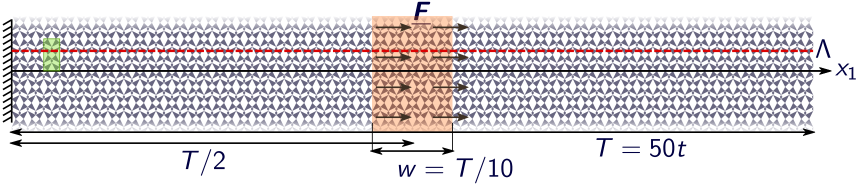
$$\boldsymbol{E} : \boldsymbol{C} : \boldsymbol{E} \propto \frac{1}{|\log \rho|}$$  $^2 + \rho^2 \Big[$  $^2 +$  $^2 \Big]$

Illustration of strain-gradient effects



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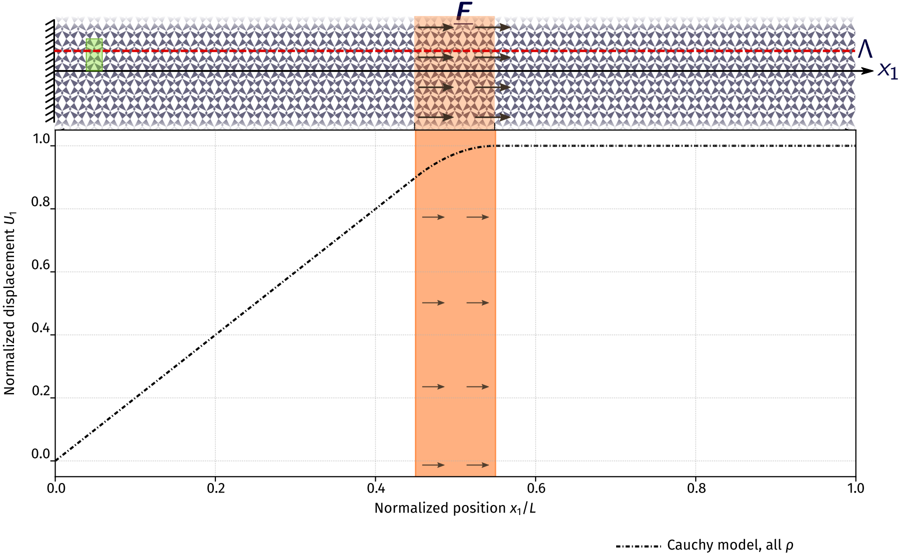

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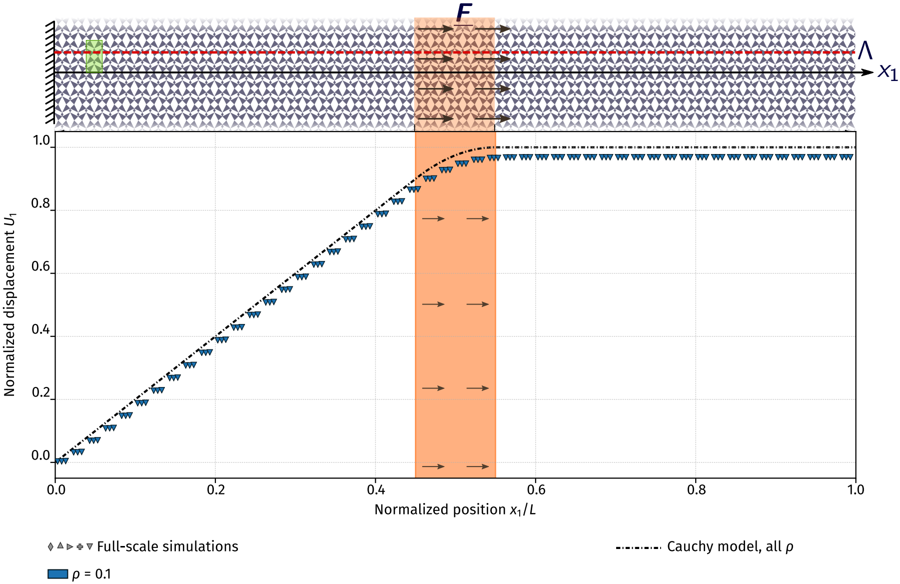

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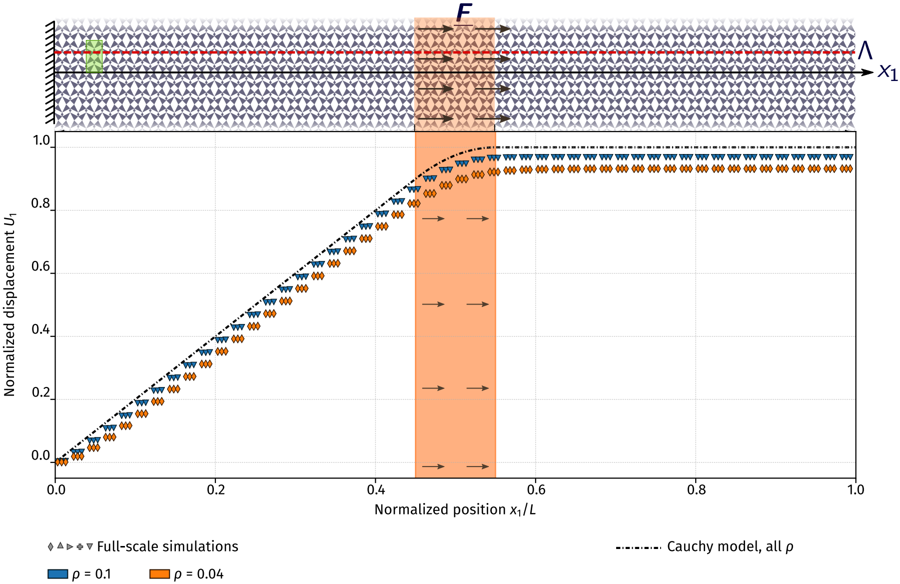

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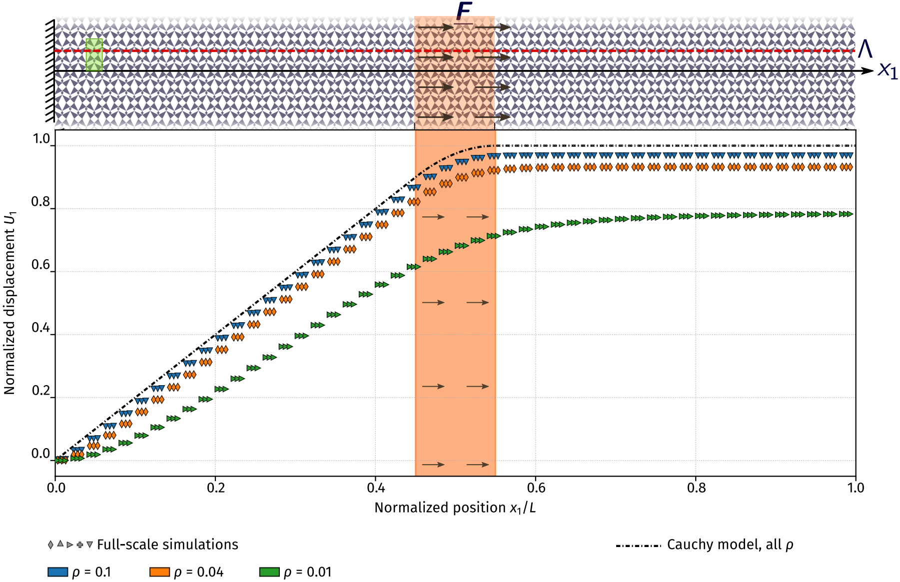

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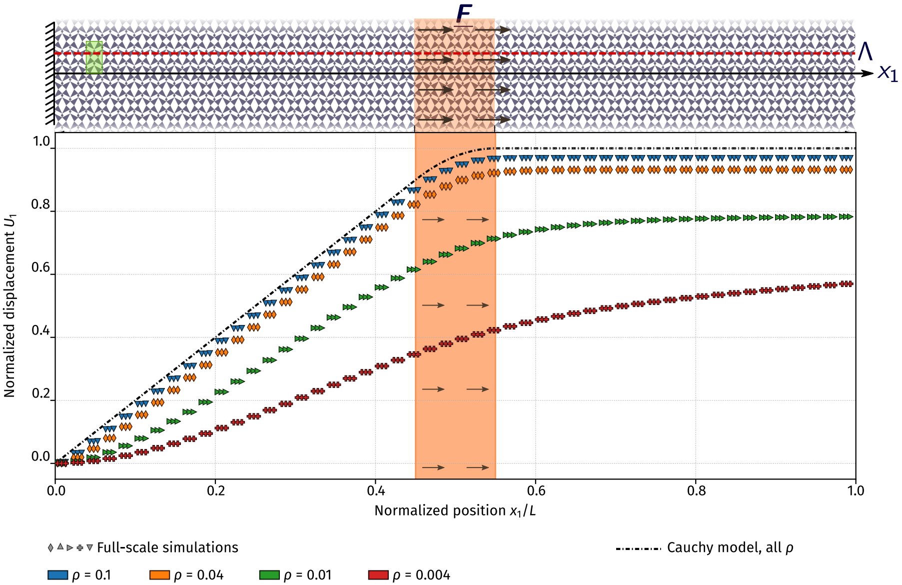

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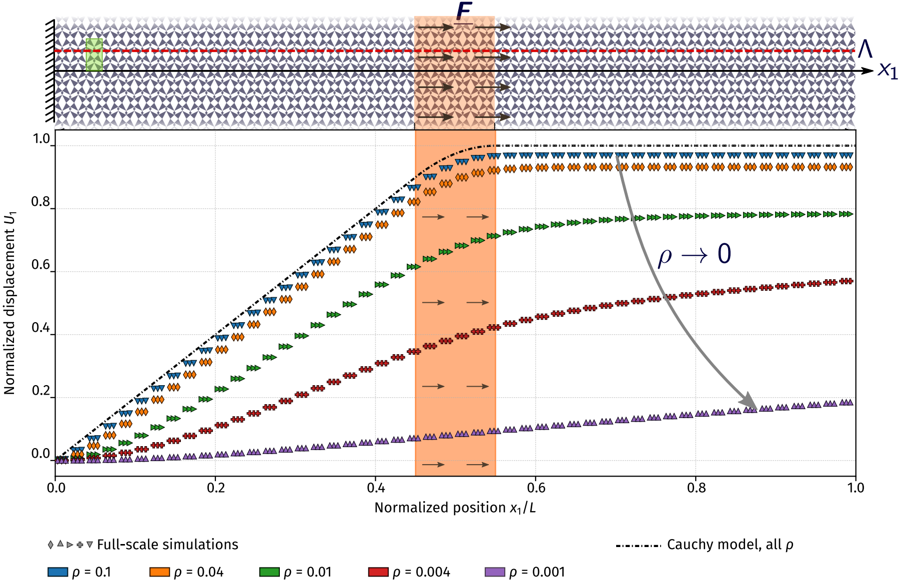

Second-order unit-cell problems

$$K_{\alpha\beta\gamma} = E_{\alpha\beta,\gamma}, \quad \underline{\boldsymbol{u}}^{\mathsf{K}}(\underline{\boldsymbol{y}}) = \frac{1}{2}\widetilde{\boldsymbol{K}} : \left(\underline{\boldsymbol{y}} \otimes \underline{\boldsymbol{y}}\right) + \eta\, \boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \left(\boldsymbol{K} \cdot \underline{\boldsymbol{y}}\right) + \eta^2\, \boldsymbol{h}^2(\underline{\boldsymbol{y}}) \vdots \boldsymbol{K},$$

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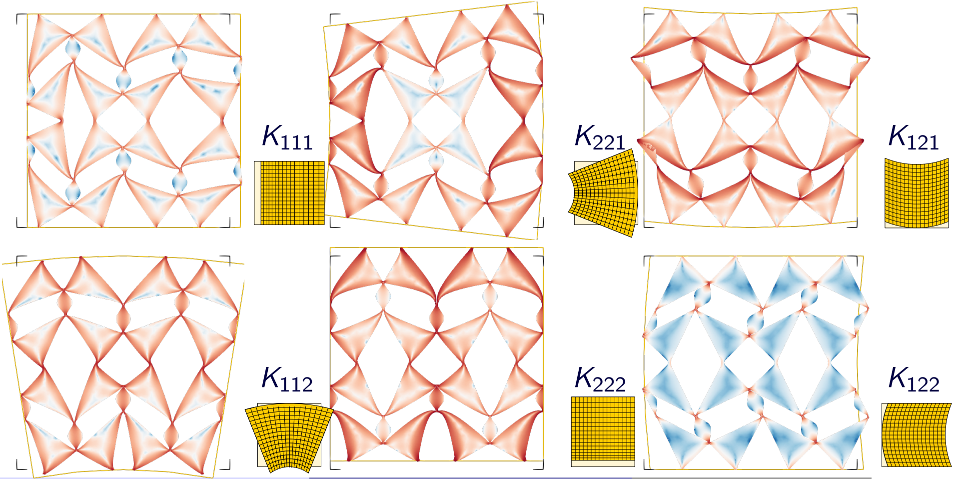

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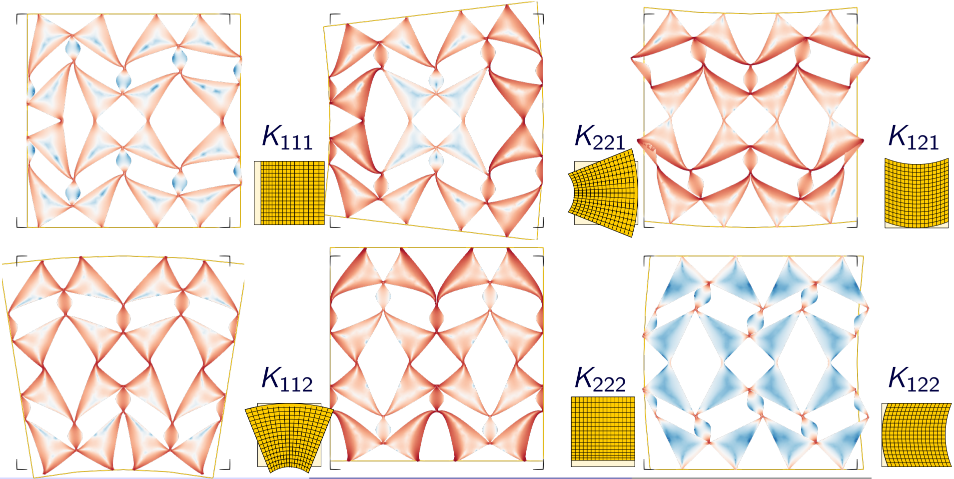

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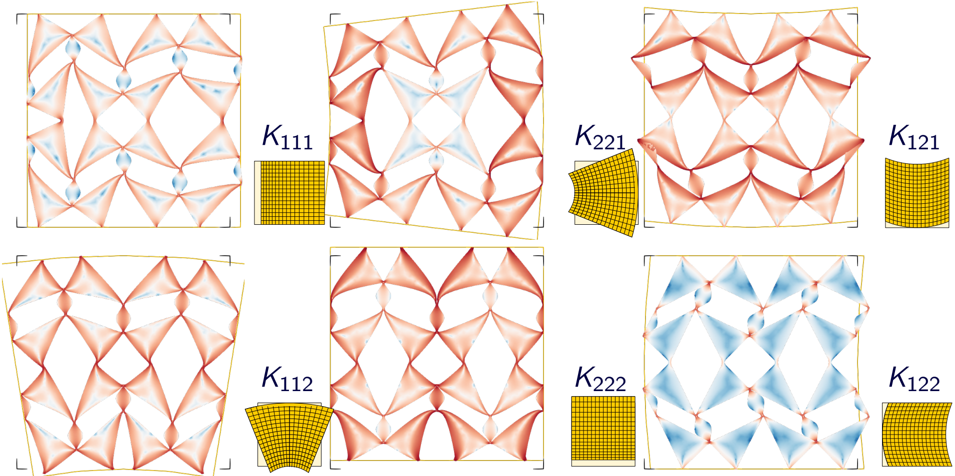

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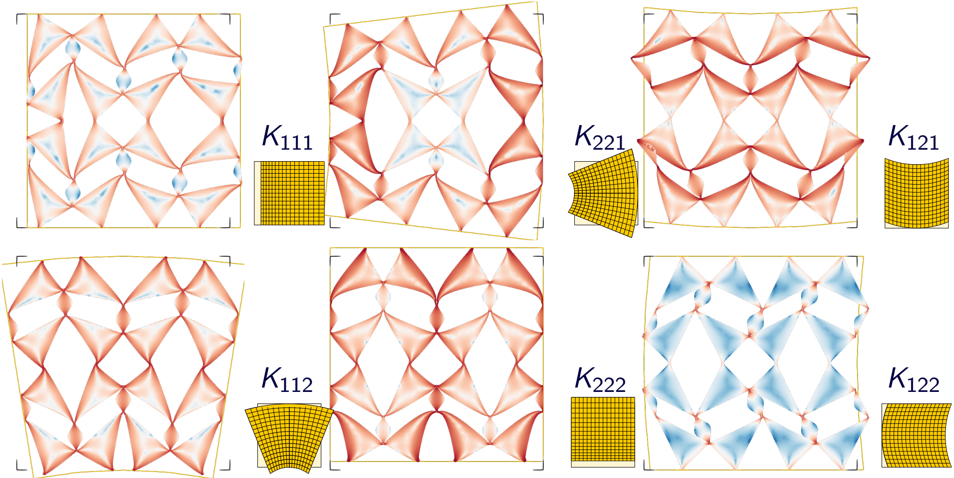

Strain-gradient homogenized energy?

2^{nd} order truncation of displacement expansion:

$$\underline{\boldsymbol{u}}^{\mathsf{K}}(\underline{\boldsymbol{y}},\underline{\boldsymbol{Y}}) = \underline{\boldsymbol{U}}(\underline{\boldsymbol{Y}}) + \eta\,\boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E}(\underline{\boldsymbol{Y}}) + \eta^2\,\boldsymbol{h}^2(\underline{\boldsymbol{y}}) \,\vdots\, \boldsymbol{K}(\underline{\boldsymbol{Y}})$$

Averaged strain energy:

$$W^\rho = \frac{1}{2}\left[\boldsymbol{E} : \boldsymbol{C}^\rho : \boldsymbol{E} + \eta^2 \boldsymbol{K}^{\mathsf{T}} \,\vdots\, \boldsymbol{D}^\rho \,\vdots\, \boldsymbol{K} + \eta^4 (\underline{\boldsymbol{\nabla}}_Y \boldsymbol{K})^{\mathsf{T}} \,\vdots\vdots\, \boldsymbol{H}^\rho \,\vdots\vdots\, (\underline{\boldsymbol{\nabla}}_Y \boldsymbol{K})\right]$$

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Caveat!

- $\boldsymbol{D}^\rho = \left\langle \left(\boldsymbol{h}^1\right)^{\mathsf{T}} \cdot \boldsymbol{c} \cdot \boldsymbol{h}^1 \right\rangle - \left\langle \left(\underline{\boldsymbol{\nabla}}^s_y \boldsymbol{h}^2\right)^{\mathsf{T}} : \boldsymbol{c} : \underline{\boldsymbol{\nabla}}^s_y \boldsymbol{h}^2 \right\rangle$

  not necessarily positive...
- $\boldsymbol{H}^\rho$ "regularizing"

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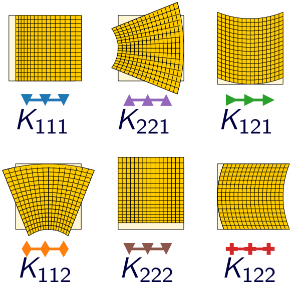
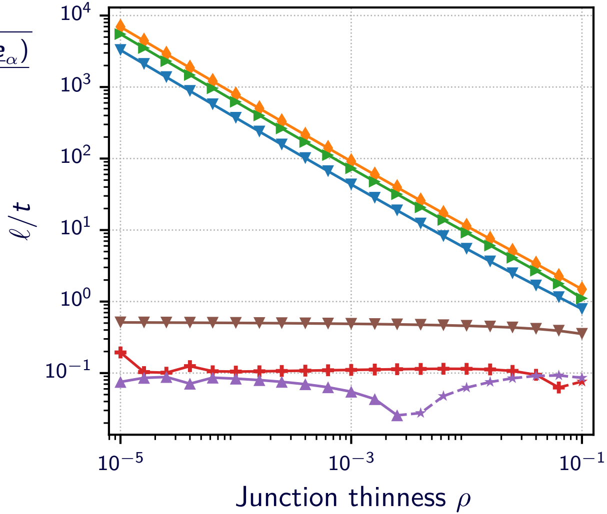
Retaining significant strain-gradient contributions

$$\ell_{i,\alpha} = \sqrt{\frac{(\underline{\boldsymbol{e}}_\alpha \otimes \boldsymbol{E}_i) \,\vdots\, \boldsymbol{D} \,\vdots\, (\boldsymbol{E}_i \otimes \underline{\boldsymbol{e}}_\alpha)}{\boldsymbol{E}_i : \boldsymbol{C} : \boldsymbol{E}_i}}$$





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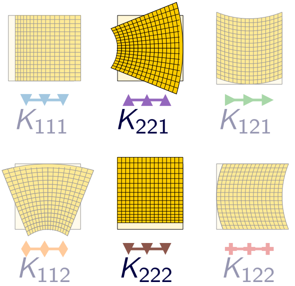

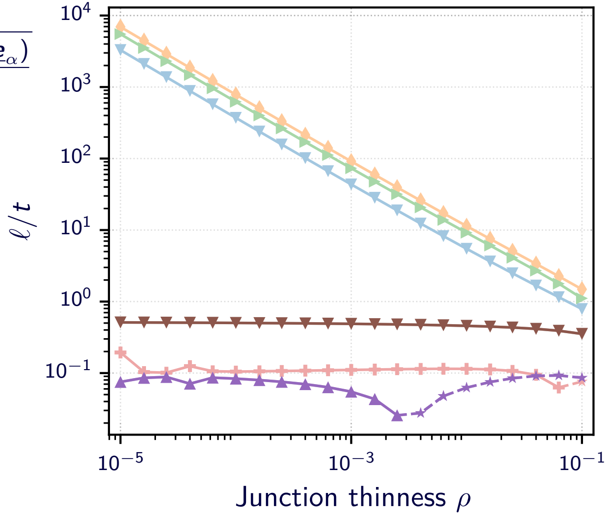

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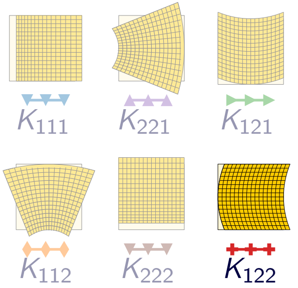
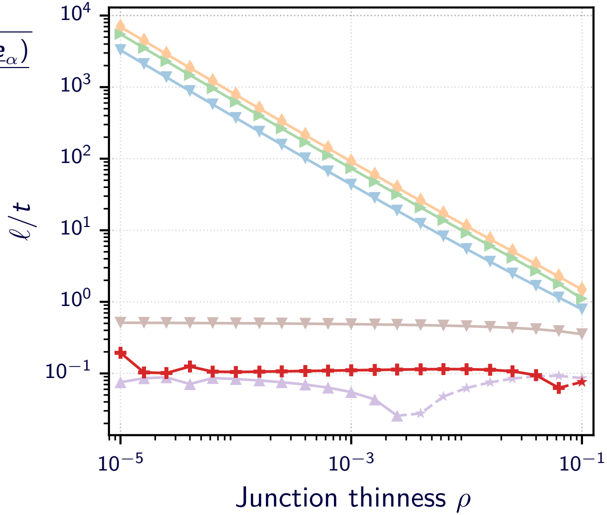
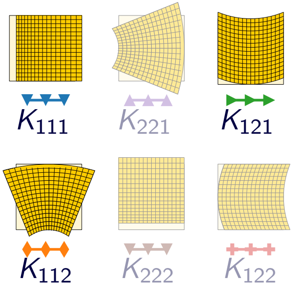
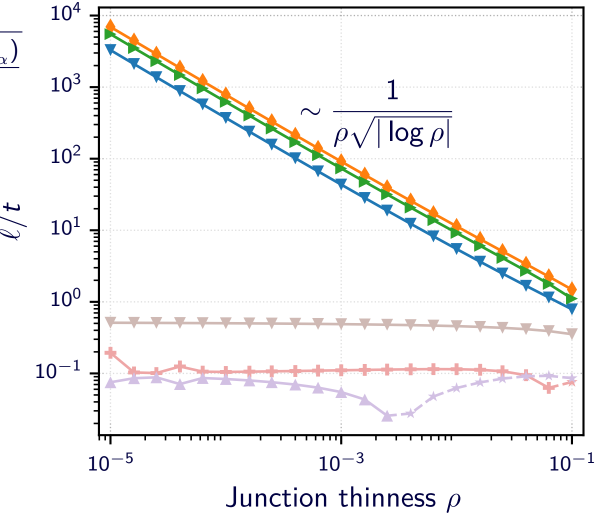




Retaining significant strain-gradient contributions

$$\ell_{i,\alpha} = \sqrt{\frac{(\underline{\boldsymbol{e}}_\alpha \otimes \boldsymbol{E}_i) \,\vdots\, \boldsymbol{D} \,\vdots\, (\boldsymbol{E}_i \otimes \underline{\boldsymbol{e}}_\alpha)}{\boldsymbol{E}_i : \boldsymbol{C} : \boldsymbol{E}_i}}$$



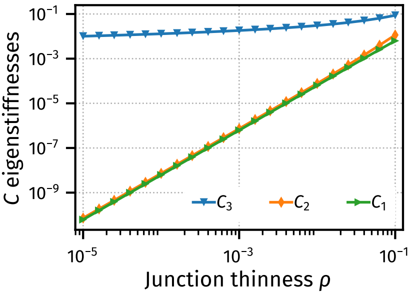


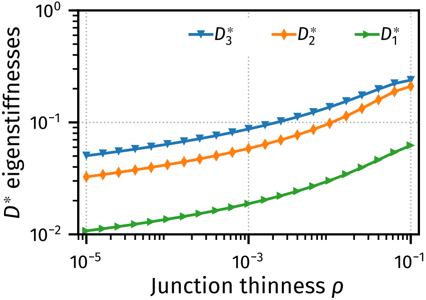
The well-posed strain-gradient homogenized model

$\boldsymbol{P}$ = projector on span $\{K_{111}, K_{112}, K_{121}\}$, $\quad \boldsymbol{D}^{*,\rho} = \boldsymbol{P} \vdots \boldsymbol{D}^{\rho} \vdots \boldsymbol{P}$

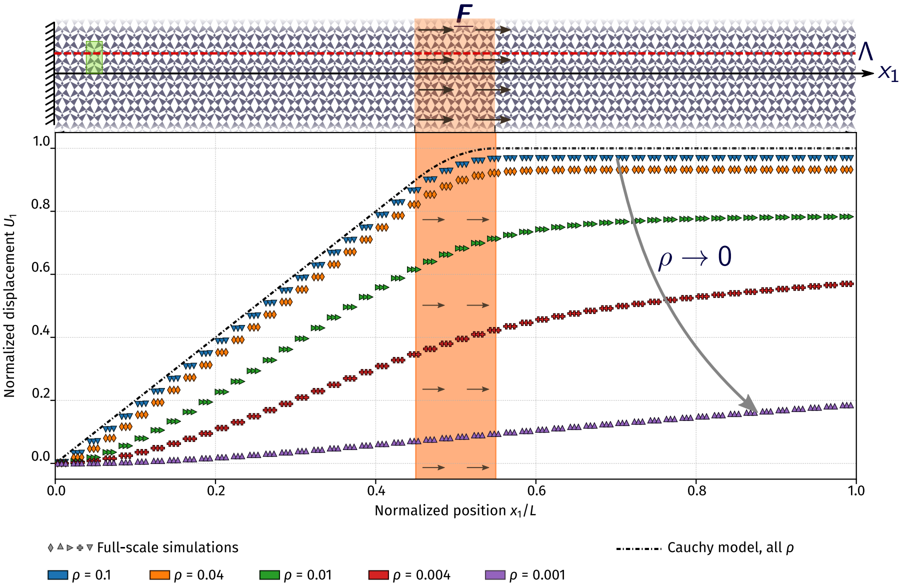
$\Rightarrow$ **Positive strain-gradient homogenized energy density**

$$W(\boldsymbol{E}, \boldsymbol{K}) = \frac{1}{2}\left(\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} + \eta^2 \boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K}\right)$$

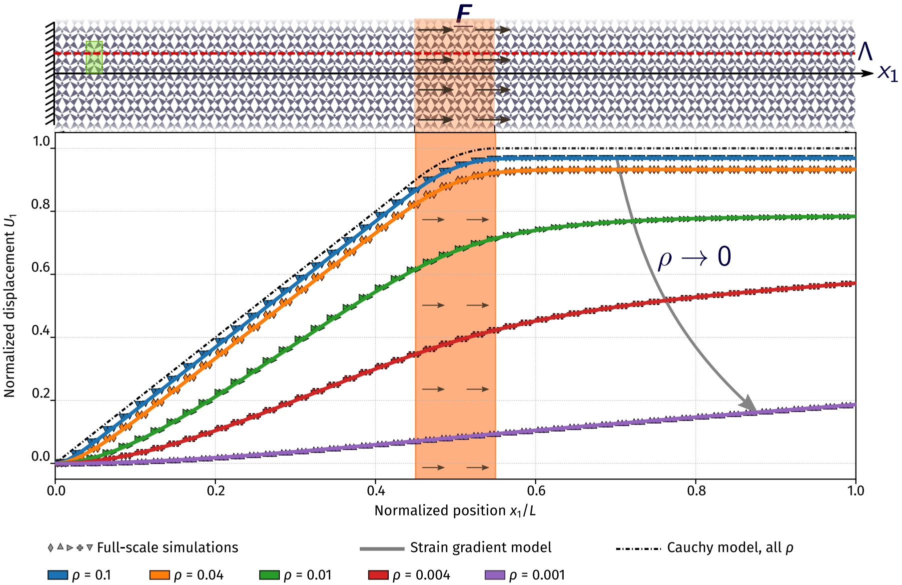




Strain-gradient homogenized model

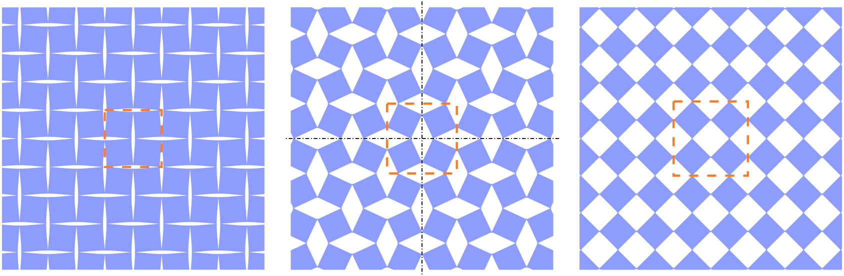


Strain-gradient homogenized model

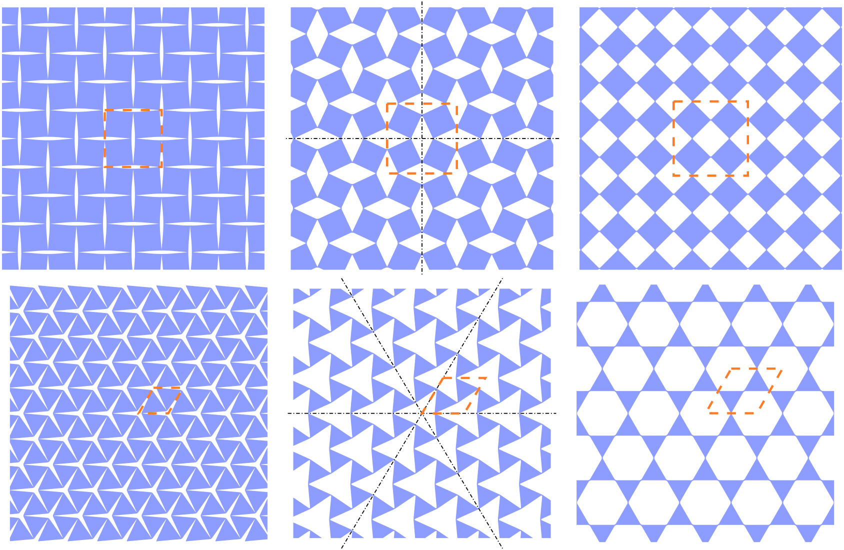


Contents

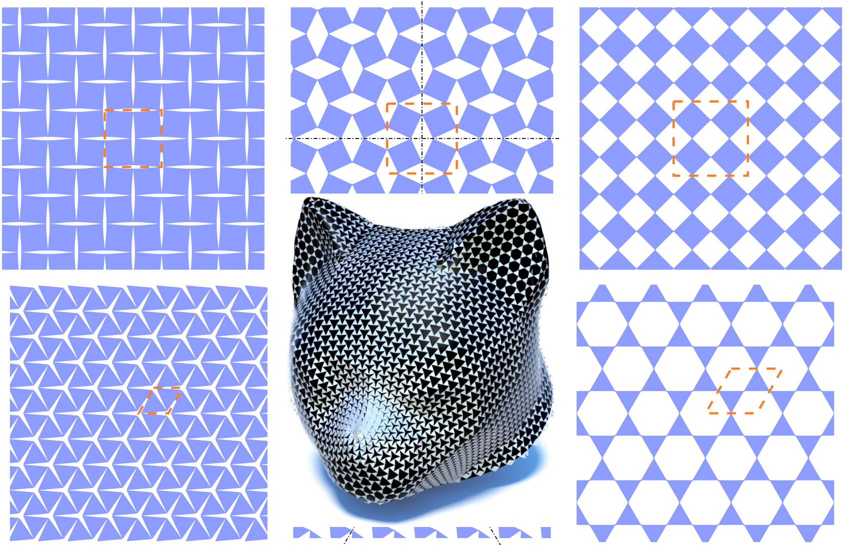
Square and triangle auxetic microstructures



Square and triangle auxetic microstructures

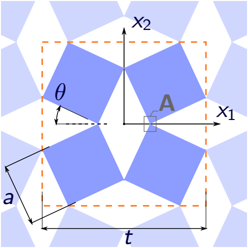


Square and triangle auxetic microstructures



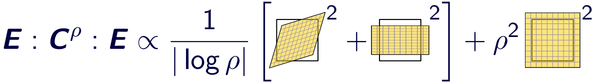
Konaković-Luković et al. (2016)

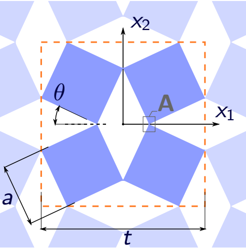
Square auxetic microstructures

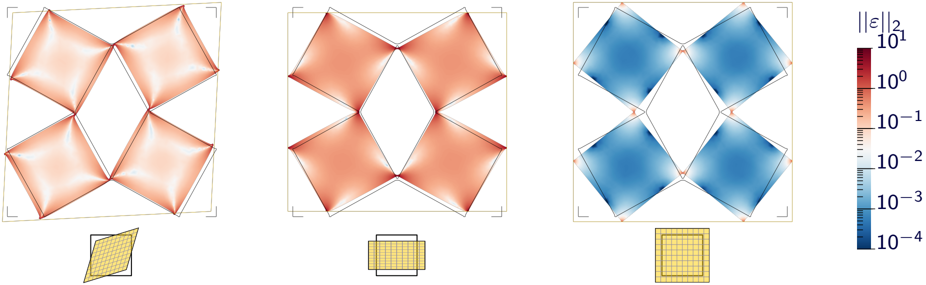


Square auxetic microstructures

First-gradient:

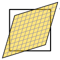
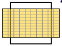
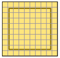




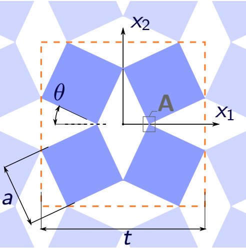


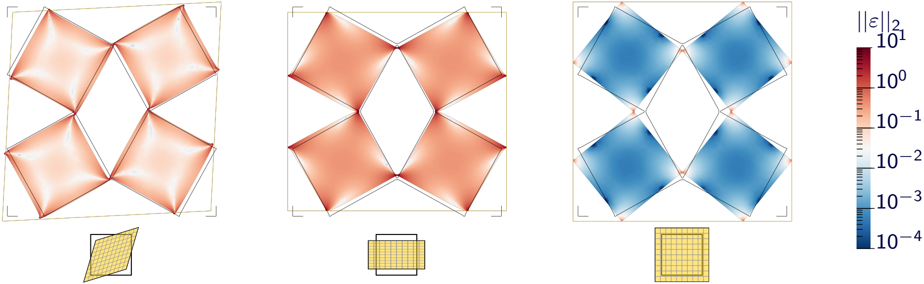
Square auxetic microstructures

First-gradient:

$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|}\Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}

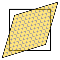
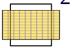
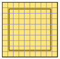
$\Rightarrow$ Dilation  is a floppy mode



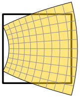
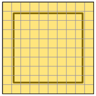
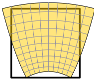
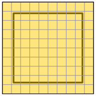


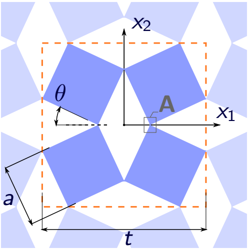
Square auxetic microstructures

First-gradient:

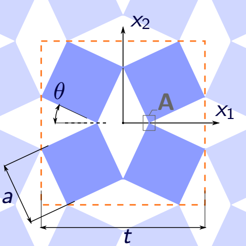
$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}

Strain-gradient:

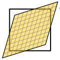
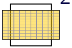
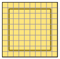
 $= \partial_1$  and  $= \partial_2$ 



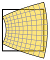
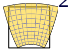
Square auxetic microstructures

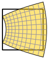


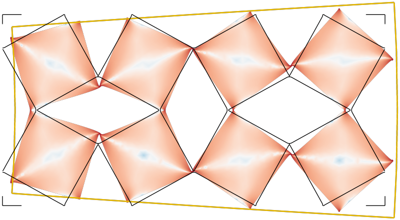
First-gradient:

$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}

Strain-gradient:

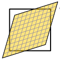
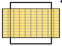
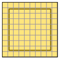
$\boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$

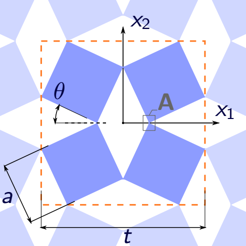




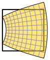
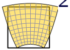
Square auxetic microstructures

First-gradient:

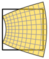
$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}

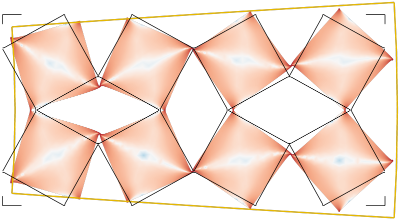


Strain-gradient:

$\boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$

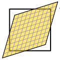
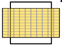
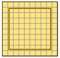
$\Rightarrow$ Isotropic strain gradient energy

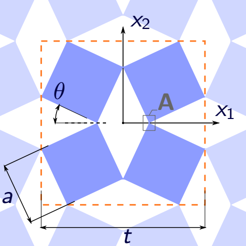




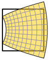
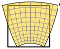
Square auxetic microstructures

First-gradient:

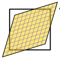
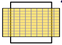
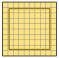
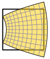
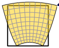
$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}



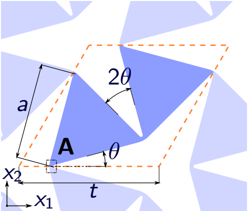
Strain-gradient:

$\boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$

Homogenized energy:

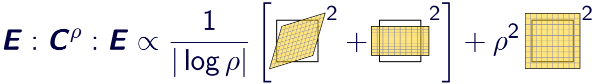
$W^{\rho} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big) + \rho^2$  $^2 + \frac{\eta^2}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$

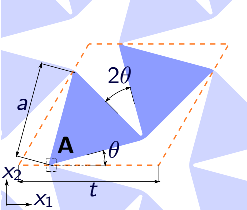
Triangle auxetic microstructures

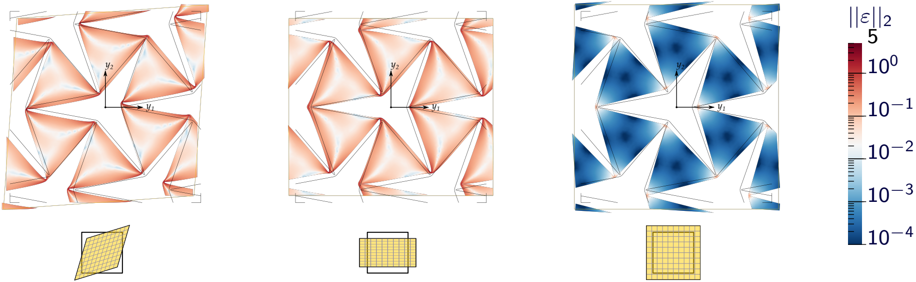


Triangle auxetic microstructures

First-gradient:

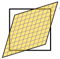
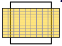
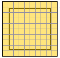


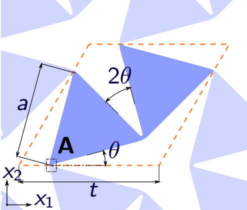




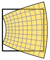
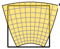
Triangle auxetic microstructures

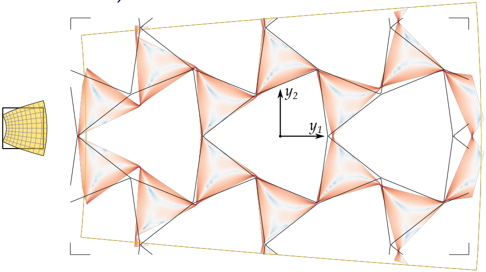
First-gradient:

$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}



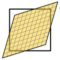
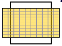
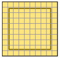
Strain-gradient:

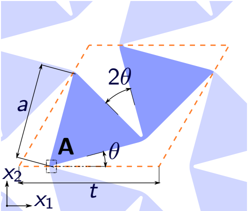
$\boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$



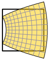
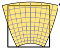
Triangle auxetic microstructures

First-gradient:

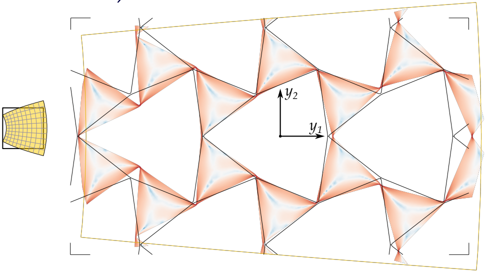
$$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \left[\right.$$  $^2 +$  $^2 \left. \right] + \rho^2$  ^{2}



Strain-gradient:

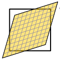
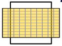
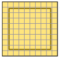
$$\boldsymbol{K}^{\mathsf{T}} \,\vdots\, \boldsymbol{D}^{*,\rho} \,\vdots\, \boldsymbol{K} \propto \frac{1}{|\log \rho|} \left(\right.$$  $^2 +$  $^2 \left. \right)$

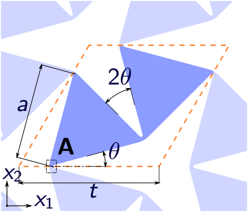
$\Rightarrow$ Isotropic strain gradient energy?



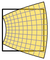
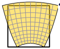
Triangle auxetic microstructures

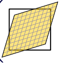
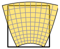
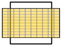
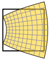
First-gradient:

$\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} \propto \frac{1}{|\log \rho|} \Big[$  $^2 +$  $^2 \Big] + \rho^2$  ^{2}

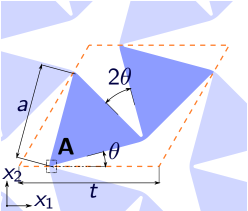


Strain-gradient:

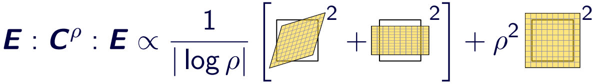
$\boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $^2 +$  $^2 \Big)$

$\boldsymbol{E} : \boldsymbol{B}^{*,\rho} \vdots \boldsymbol{K} \propto \frac{1}{|\log \rho|} \Big($  $\times$  $+$  $\times$  $\Big)$

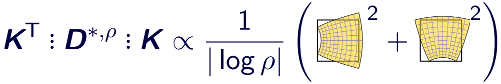
Triangle auxetic microstructures



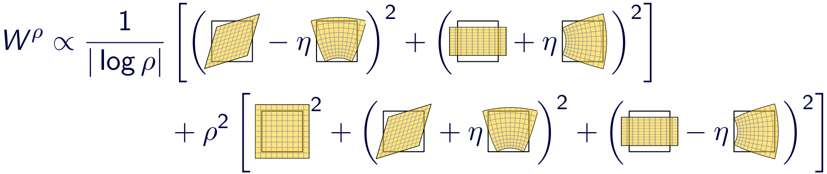
First-gradient:



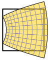
Strain-gradient:

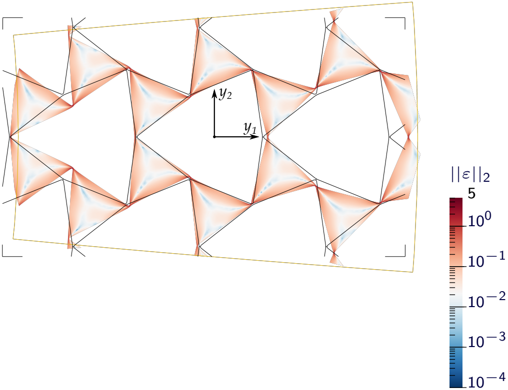


Homogenized energy:

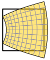


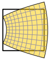
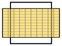
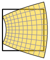
Effect of microadjustement on the local fields

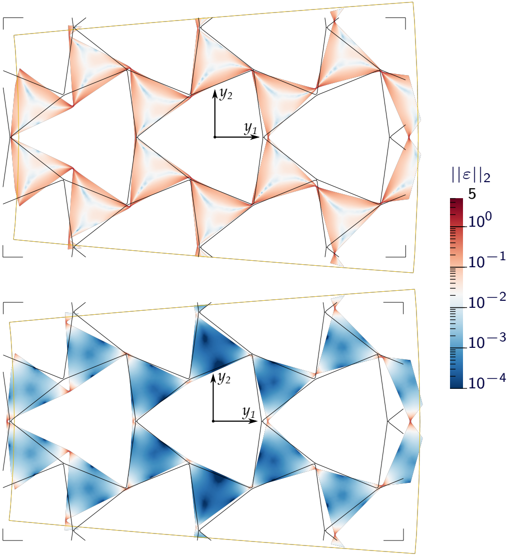
Only 



Effect of microadjustement on the local fields

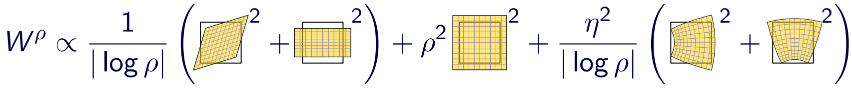
Only 

 with  $= \eta$ 

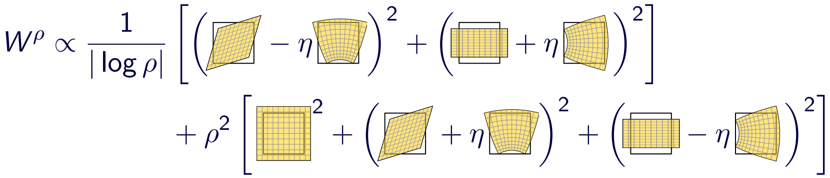


Square and triangle auxetic microstructures

Squares:



Triangles:



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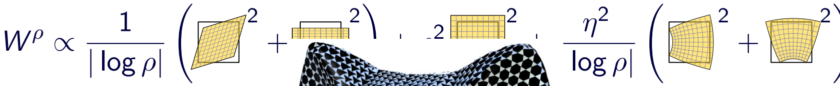
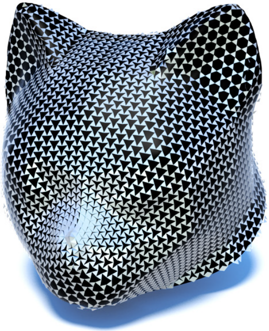
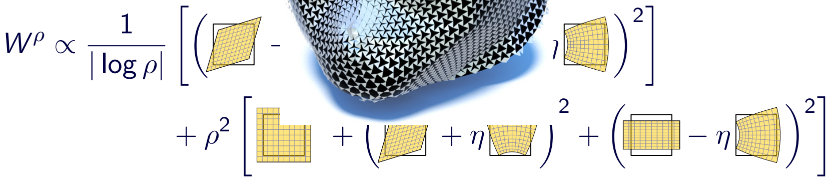




Triangles:

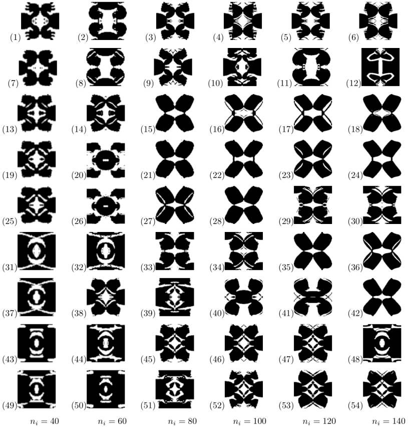


Contents

Topology optimisation?

- Stiff and soft phases:

  $\gamma = E_{\min}/E_{\max} = 0.01$



Topology optimisation?

- Stiff and soft phases:

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  Sokolowski and Zochowski (1999)
  Amstutz et al. (2010)
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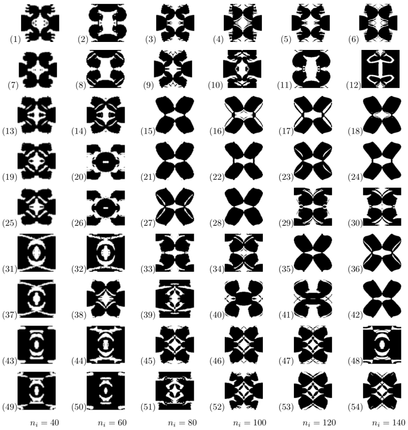

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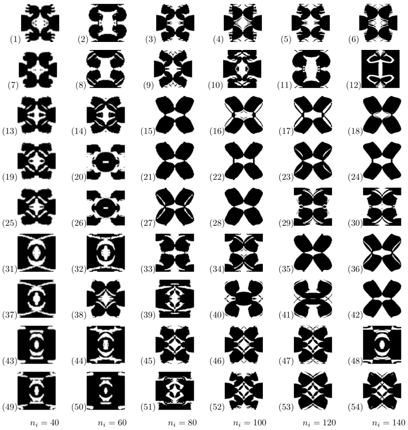
$$\max \left\{ \frac{(\underline{\boldsymbol{e}}_\alpha \otimes \boldsymbol{E}_i) \vdots \boldsymbol{D} \vdots (\boldsymbol{E}_i \otimes \underline{\boldsymbol{e}}_\alpha)}{\boldsymbol{E}_i : \boldsymbol{C} : \boldsymbol{E}_i} \right\}$$



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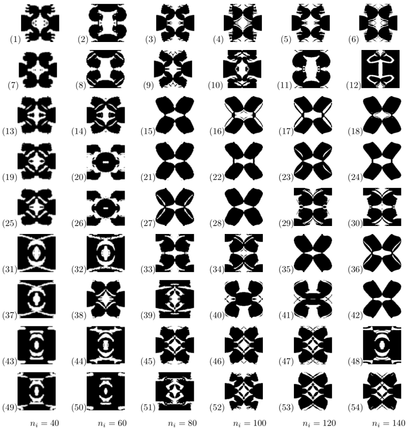
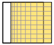
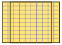
- The volume fraction is let free



ℓ_{111}

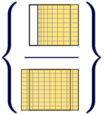
$$\max\left\{\frac{D_{111111}}{C_{1111}}\right\} = \max\left\{\frac{\square}{\square}\right\}$$

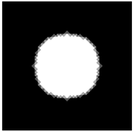




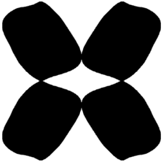
ℓ_{111}

$$\max\left\{\frac{D_{111111}}{C_{1111}}\right\} = \max$$

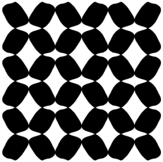




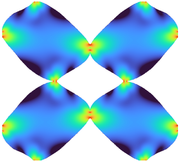
Initialization

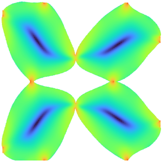


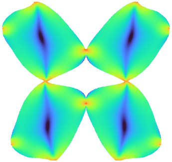
Unit cell

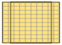


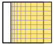
Nine unit cells

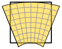






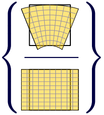
E_{11} 

K_{111} 

K_{112} 

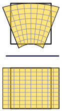
ℓ_{112}

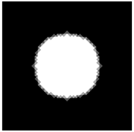
$$\max\left\{\frac{D_{112211}}{C_{1111}}\right\} = \max$$



ℓ_{112}

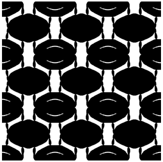
$$\max\left\{\frac{D_{112211}}{C_{1111}}\right\} = \max\left\{\frac{\square}{\square}\right\}$$



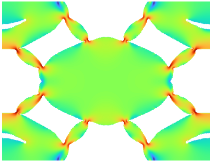


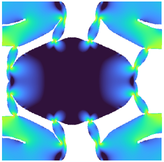
Initialization

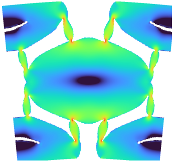
Unit cell

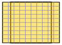


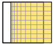
Nine unit cells

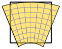






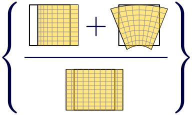
E_{11} 

K_{111} 

K_{112} 

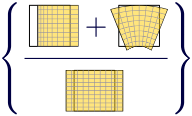
$\ell_{111} + \ell_{112}$

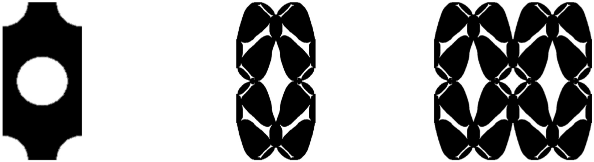
$$\max\left\{\frac{D_{111111}+D_{112211}}{C_{1111}}\right\} = \max$$



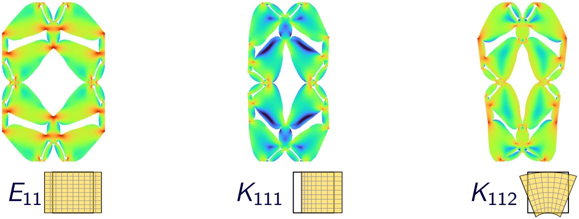
$\ell_{111} + \ell_{112}$

$$\max \left\{ \frac{D_{111111} + D_{112211}}{C_{1111}} \right\} = \max$$



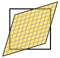


Initialization | Unit cell | Nine unit cells



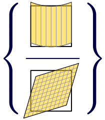
ℓ_{121}

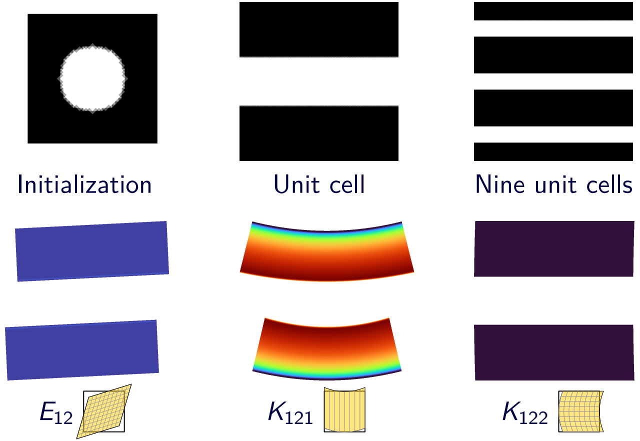
$$\max\left\{\frac{D_{121121}}{C_{1212}}\right\} = \max\left\{\frac{\quad}{\quad}\right\}$$



ℓ_{121}

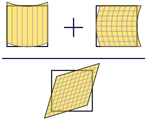
$$\max\left\{\frac{D_{121121}}{C_{1212}}\right\} = \max$$





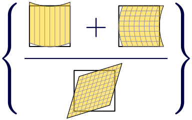
$\ell_{121} + \ell_{122}$

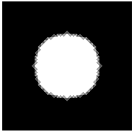
$$\max\left\{\frac{D_{121121} + D_{122221}}{C_{1212}}\right\} = \max\left\{\ \cdots\ \right\}$$



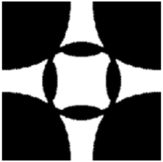
$\ell_{121} + \ell_{122}$

$$\max \left\{ \frac{D_{121121} + D_{122221}}{C_{1212}} \right\} = \max$$

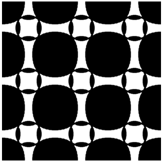




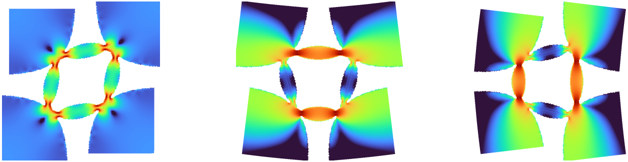
Initialization

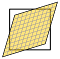
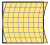


Unit cell



Nine unit cells



E_{12}  K_{121}  K_{122} 

Conclusion

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- The asymptotic expansion yields predictive strain-gradient moduli *when they are significant*
- New designs from topology optimization

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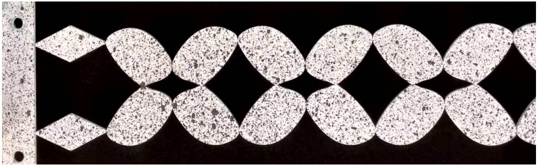

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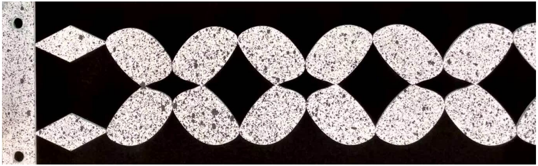

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Outlooks

- Enriched continua?
- Finite deformations?

"Know that, for Adolph, myself and Mechanics - to which we devote our lives - this is a great moment!..."

"Le Bandit Manchot", Lucky Luke (1981), Bob de Groot - Morris, Dargaud pub.

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