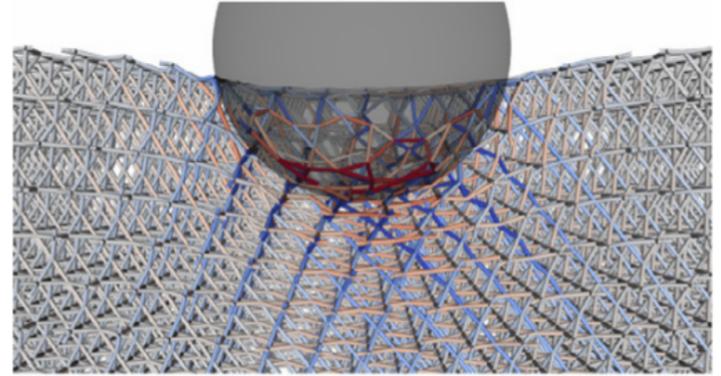
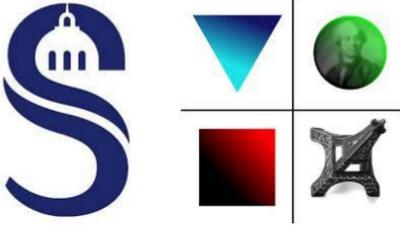


Is higher-order homogenization useful?

CLAIRE LESTRINGANT

Institut ∂ 'Alembert

Sorbonne Université

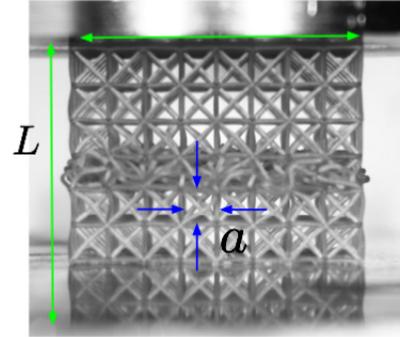
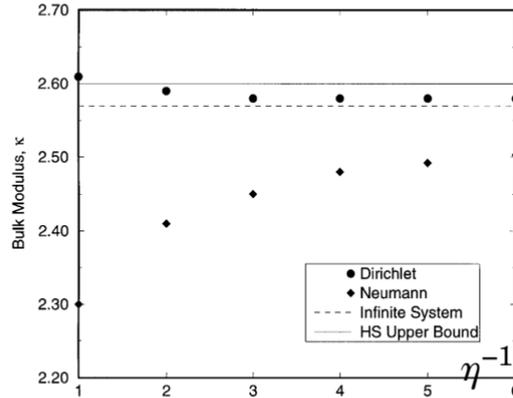
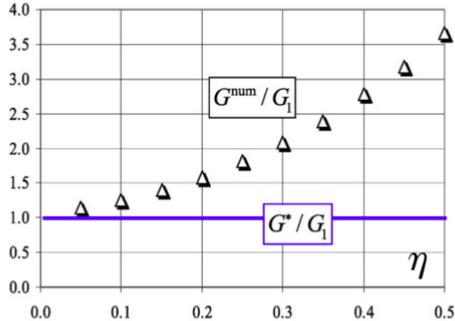
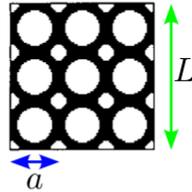
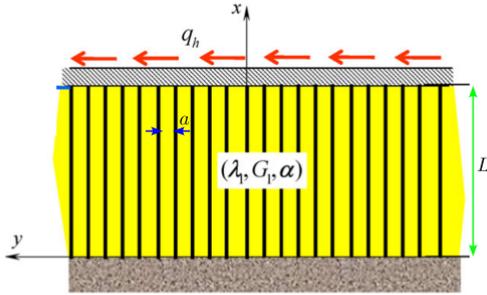


[Phlipot+Kochmann, 2019]

Colloque Mécamat, Aussois, 27-31 janvier 2025

Scale effects in *elastic* periodic materials

Scale effects due to high contrast, poor scale separation ($\eta = \frac{a}{L}$ not so small), boundary effects



[de Buhan+Hassen 2007]

[Pecullan et al '98]

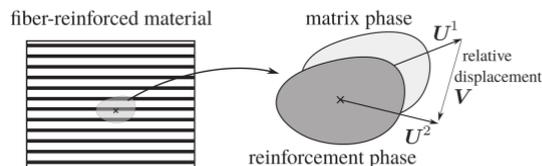
[Dias 2019]

Non-local effective behaviour

- generalized continua [Toupin '62] [Mindlin '64] [Forest 2006]
 - additional dofs: micromorphic, Cosserat, micropolar
 - **higher-order gradients**
- multiphase models [de Buhan+Hassen 2007] [Bleyer 2018]
- other non-local

In practice:

- how to choose the relevant theory for a given microstructure?
 - how to determine the effective parameters?
- Intuition, trial-and-error, identification procedures



- Asymptotic homogenization: rigorous and explicit way to derive the generalized model from the microstructure [Sanchez-Palencia '80] [Bakhvalov+Panasenکو '84] [Boutin '96, 2020]
- In this talk: gradient models for *linear* elastostatics

- **Asymptotic homogenization for linear elastostatics**
- A sign problem
- Is it worth the effort?

Microscopic model and two-scale Ansatz

5/42

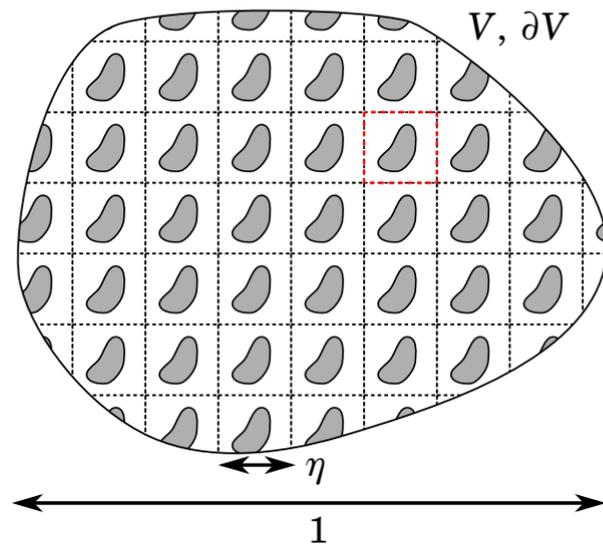
For sample V of a periodic material with unit cell $\eta \Omega$
total energy (with distributed force \mathbf{f})

$$\Phi[\boldsymbol{\xi}] = \int_V \frac{1}{2} \boldsymbol{\varepsilon}(\mathbf{x}) : \mathbf{c}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \, d\mathbf{x} - \int_V \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\xi}(\mathbf{x}) \, d\mathbf{x}$$

with $\boldsymbol{\varepsilon} = \nabla \boldsymbol{\xi}$

equilibrium problem (strong form)

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}, & \boldsymbol{\sigma} = \mathbf{c} : \boldsymbol{\varepsilon} & \text{on } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} & & \text{on } \partial V \end{cases}$$



Microscopic model and two-scale Ansatz

6/42

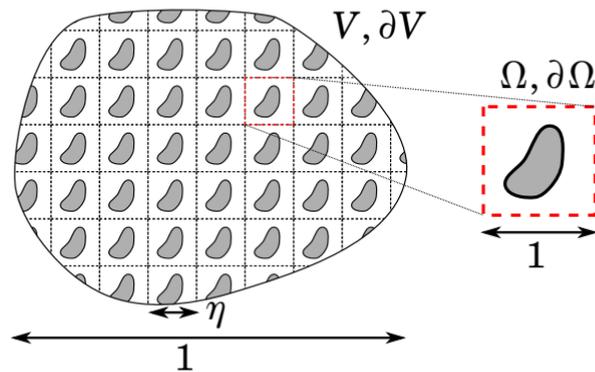
For sample V of a periodic material with unit cell $\eta \Omega$
total energy (with distributed force \mathbf{f})

$$\Phi[\boldsymbol{\xi}] = \int_V \frac{1}{2} \boldsymbol{\varepsilon}(\mathbf{x}) : \mathbf{c}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \, d\mathbf{x} - \int_V \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\xi}(\mathbf{x}) \, d\mathbf{x}$$

$$\text{with } \boldsymbol{\varepsilon} = \nabla \boldsymbol{\xi}$$

equilibrium problem (strong form)

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = 0, & \boldsymbol{\sigma} = \mathbf{c} : \boldsymbol{\varepsilon} & \text{on } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} & & \text{on } \partial V \end{cases}$$



with $\eta \ll 1$, introduce rescaled variable

$$\mathbf{y} = \eta^{-1} \mathbf{x} \quad \text{with} \quad \eta \ll 1$$

assume displacement is a function of the two variables (\mathbf{x}, \mathbf{y})

$$\boldsymbol{\xi}(\mathbf{x}, \eta^{-1} \mathbf{x})$$

Ansatz for the displacement

$$\xi(\mathbf{x}, \mathbf{y}) = \mathbf{U}(\mathbf{x}) + \eta \mathbf{Y}(\mathbf{x}, \mathbf{y}) \quad \text{with} \quad \langle \mathbf{Y} \rangle(\mathbf{x}) = \mathbf{0}, \quad \text{where} \quad \langle \cdot \rangle = \int_{\Omega} \cdot \, d\mathbf{y}.$$

$\mathbf{U}(\mathbf{x})$ is the macroscopic displacement and $\mathbf{Y}(\mathbf{x}, \mathbf{y})$ is the microscopic shift (periodic in \mathbf{y}), with

$$\mathbf{U}(\mathbf{x}) = \sum_{k=0}^{\infty} \eta^k \mathbf{U}_k(\mathbf{x}), \quad \mathbf{Y}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{\infty} \eta^k \mathbf{Y}_k(\mathbf{x}, \mathbf{y})$$

Energy & equilibrium problem, with $\nabla = \nabla_{\mathbf{x}} + \eta^{-1} \nabla_{\mathbf{y}}$

$$\Phi[\mathbf{U}, \mathbf{Y}] = \int_V \frac{1}{2} \langle (\nabla_{\mathbf{x}} \mathbf{U} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}) : \mathbf{c}(\mathbf{y}) : (\nabla_{\mathbf{x}} \mathbf{U} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}) \rangle \, d\mathbf{x} - \int_V \mathbf{f} \cdot \mathbf{U} \, d\mathbf{x}$$

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}, & \boldsymbol{\sigma} = \mathbf{c} : (\nabla_{\mathbf{x}} \mathbf{U} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}) & \text{on } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} & \text{on } \partial V \end{cases}$$

→ Energy: starting point for a **variational homogenization procedure** [Berdichevski '79, 2009] [Lions, '81] [Allaire+Ganaoui, 2009] [Le+Marigo, 2018] [Audoly+CL, 2023]

1. Solve variational *cell problem* for \mathbf{y} -periodic micro shift \mathbf{Y}

$$\mathbf{D}_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g}(\mathbf{x}) \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = 0 \quad \forall \delta \mathbf{Y} \quad \mathbf{y}\text{-periodic}$$

→. solution: $\mathbf{Y} = \mathbf{Y}^*(\mathbf{E}, \mathbf{y})$ with $\mathbf{E} = \nabla_{\mathbf{x}} \mathbf{U}$

1. Solve variational *cell problem* for \mathbf{y} -periodic micro shift \mathbf{Y}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g}(\mathbf{x}) \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = 0 \quad \forall \delta \mathbf{Y} \quad \mathbf{y}\text{-periodic}$$

→. solution: $\mathbf{Y} = \mathbf{Y}^*(\mathbf{E}, \mathbf{y})$ with $\mathbf{E} = \nabla_{\mathbf{x}} \mathbf{U}$

2. Compute effective energy in terms of \mathbf{U} only as $\Phi^*[\mathbf{U}] = \Phi[\mathbf{U}, \mathbf{Y}^*(\mathbf{E})]$

$$\Phi^*[\mathbf{U}] = \int_V (\langle \boldsymbol{\sigma}^* : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}, \mathbf{y})) \rangle - \mathbf{f} \mathbf{U}) d\mathbf{x}$$

$$\text{with } \boldsymbol{\sigma}^*(\mathbf{x}, \mathbf{y}) = \mathbf{c} : (\mathbf{E}(\mathbf{x}) + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}(\mathbf{x}), \mathbf{y}))$$

1. Solve variational *cell problem* for \mathbf{y} -periodic micro shift \mathbf{Y}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g}(\mathbf{x}) \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = 0 \quad \forall \delta \mathbf{Y} \quad \mathbf{y}\text{-periodic}$$

→. solution: $\mathbf{Y} = \mathbf{Y}^*(\mathbf{E}, \mathbf{y})$ with $\mathbf{E} = \nabla_{\mathbf{x}} \mathbf{U}$

2. Compute effective energy in terms of \mathbf{U} only as $\Phi^*[\mathbf{U}] = \Phi[\mathbf{U}, \mathbf{Y}^*(\mathbf{E})]$

$$\Phi^*[\mathbf{U}] = \int_V (\langle \boldsymbol{\sigma}^* : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}, \mathbf{y})) \rangle - \mathbf{f} \mathbf{U}) d\mathbf{x}$$

with $\boldsymbol{\sigma}^*(\mathbf{x}, \mathbf{y}) = \mathbf{c} : (\mathbf{E}(\mathbf{x}) + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}(\mathbf{x}), \mathbf{y}))$

3. Formulate *effective* equilibrium problem

$$D_{\mathbf{U}} \Phi^*[\mathbf{U}, \delta \mathbf{U}] = \int_V (\langle \boldsymbol{\sigma}^* \rangle : \nabla_{\mathbf{x}} \delta \mathbf{U} - \mathbf{f} \delta \mathbf{U}) d\mathbf{x} + D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}^*(\mathbf{E}); \mathbf{Y}^*(\nabla_{\mathbf{x}} \delta \mathbf{U})] = 0 \quad \forall \delta \mathbf{U}$$

1. Solve variational *cell problem* for \mathbf{y} -periodic micro shift \mathbf{Y}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g}(\mathbf{x}) \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = 0 \quad \forall \delta \mathbf{Y} \quad \mathbf{y}\text{-periodic}$$

→. solution: $\mathbf{Y} = \mathbf{Y}^*(\mathbf{E}, \mathbf{y})$ with $\mathbf{E} = \nabla_{\mathbf{x}} \mathbf{U}$

2. Compute effective energy in terms of \mathbf{U} only as $\Phi^*[\mathbf{U}] = \Phi[\mathbf{U}, \mathbf{Y}^*(\mathbf{E})]$

$$\Phi^*[\mathbf{U}] = \int_V (\langle \boldsymbol{\sigma}^* : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}, \mathbf{y})) \rangle - \mathbf{f} \mathbf{U}) d\mathbf{x}$$

$$\text{with } \boldsymbol{\sigma}^*(\mathbf{x}, \mathbf{y}) = \mathbf{c} : (\mathbf{E}(\mathbf{x}) + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*(\mathbf{E}(\mathbf{x}), \mathbf{y}))$$

3. Formulate *effective* equilibrium problem

$$D_{\mathbf{U}} \Phi^*[\mathbf{U}, \delta \mathbf{U}] = \int_V (\langle \boldsymbol{\sigma}^* \rangle : \nabla_{\mathbf{x}} \delta \mathbf{U} - \mathbf{f} \delta \mathbf{U}) d\mathbf{x} + \underbrace{D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}^*(\mathbf{E}); \mathbf{Y}^*(\nabla_{\mathbf{x}} \delta \mathbf{U})]}_{=0} = 0 \quad \forall \delta \mathbf{U}$$

$\text{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}^* \rangle + \mathbf{f} = \mathbf{0}$ on V + boundary conditions

1. Formulation of cell problem

Find \mathbf{y} -periodic micro shift \mathbf{Y}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g} \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = \int_V \langle \boldsymbol{\sigma} : (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \delta \mathbf{Y} \rangle d\mathbf{x} - \int_V \mathbf{g} \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x}$$

integration by parts with respect to \mathbf{x}

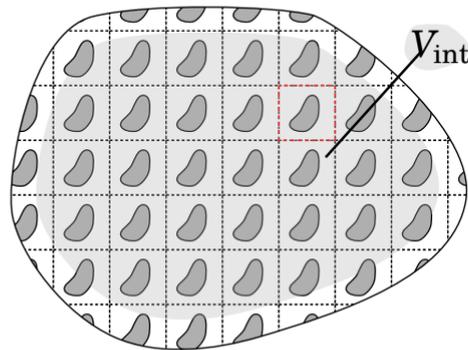
$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] = \int_V \langle -\eta \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \cdot \delta \mathbf{Y} + \boldsymbol{\sigma} : \nabla_{\mathbf{y}} \delta \mathbf{Y} \rangle d\mathbf{x} + \int_{\partial V} \langle \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \delta \mathbf{Y} \rangle ds$$

dropping boundary terms (more on this later)

$$\int_{V_{\text{int}}} \langle -(\eta \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} + \mathbf{g}) \cdot \delta \mathbf{Y} + \boldsymbol{\sigma} : \nabla_{\mathbf{y}} \delta \mathbf{Y} \rangle d\mathbf{x} = 0$$

taking constant $\delta \mathbf{Y}$ yields Lagrange multiplier

$$\mathbf{g} = -\eta \langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \rangle$$



1. Formulation of cell problem

Find \mathbf{y} -periodic micro shift \mathbf{Y}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] - \int_V \mathbf{g} \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x} = \int_V \langle \boldsymbol{\sigma} : (\eta \nabla_{\mathbf{x}}^s + \nabla_{\mathbf{y}}^s) \delta \mathbf{Y} \rangle d\mathbf{x} - \int_V \mathbf{g} \cdot \langle \delta \mathbf{Y} \rangle d\mathbf{x}$$

integration by parts with respect to \mathbf{x}

$$D_{\mathbf{Y}} \Phi[\mathbf{U}, \mathbf{Y}; \delta \mathbf{Y}] = \int_V \langle -\eta \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \cdot \delta \mathbf{Y} + \boldsymbol{\sigma} : \nabla_{\mathbf{y}} \delta \mathbf{Y} \rangle d\mathbf{x} + \int_{\partial V} \langle \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \delta \mathbf{Y} \rangle ds$$

we obtain cell problem as

$$\int_{V_{\text{int}}} \langle -\eta (\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} - \langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \rangle) \cdot \delta \mathbf{Y} + \boldsymbol{\sigma} : \nabla_{\mathbf{y}} \delta \mathbf{Y} \rangle d\mathbf{x} = 0, \forall \delta \mathbf{Y}$$

Last integration by parts w.r. to \mathbf{y} yields the cell problem in strong form

Find \mathbf{y} -periodic micro shift \mathbf{Y} , with $\langle \mathbf{Y} \rangle = 0$,

$$\begin{cases} \operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma} = \eta (\langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \rangle - \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}) & \text{on } \Omega \\ \boldsymbol{\sigma} = \mathbf{c} : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}) \end{cases}$$

1. Solving cell problem order by order

Cell problem in strong form

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma} = \eta (\langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \rangle - \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}) \quad \text{with} \quad \boldsymbol{\sigma} = \mathbf{c} : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y})$$

Plugging-in series expansions $\mathbf{E} = \sum_k \eta^k \mathbf{E}_k(\mathbf{x})$, $\mathbf{Y} = \sum_k \eta^k \mathbf{Y}_k$ yields

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_0 = \mathbf{0} \quad \text{with} \quad \boldsymbol{\sigma}_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_{\mathbf{y}} \mathbf{Y}_0)$$

solution writes

$$\mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = \boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}_0(\mathbf{x})$$

1. Solving cell problem order by order

Cell problem in strong form

$$\operatorname{div}_y \sigma = \eta (\langle \operatorname{div}_x \sigma \rangle - \operatorname{div}_x \sigma) \quad \text{with} \quad \sigma = \mathbf{c} : (\mathbf{E} + (\eta \nabla_x + \nabla_y) \mathbf{Y})$$

Plugging-in series expansions $\mathbf{E} = \sum_k \eta^k \mathbf{E}_k(\mathbf{x})$, $\mathbf{Y} = \sum_k \eta^k \mathbf{Y}_k$ yields

$$\begin{aligned} \operatorname{div}_y \sigma_0 &= \mathbf{0} & \text{with} & \quad \sigma_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_y \mathbf{Y}_0) \\ \operatorname{div}_y \sigma_1 &= \langle \operatorname{div}_x \sigma_0 \rangle - \operatorname{div}_x \sigma_0 & \text{with} & \quad \sigma_1 = \mathbf{c} : (\mathbf{E}_1 + \nabla_x \mathbf{Y}_0 + \nabla_y \mathbf{Y}_1) \quad \mathbf{Y}_0 = \Psi_0 : \mathbf{E}_0, \sigma_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_y \mathbf{Y}_0) \end{aligned}$$

solution writes

$$\mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = \Psi_0(\mathbf{y}) : \mathbf{E}_0(\mathbf{x}) + \eta (\Psi_0(\mathbf{y}) : \mathbf{E}_1(\mathbf{x}) + \Psi_1(\mathbf{y}) : \nabla \mathbf{E}_0(\mathbf{x}))$$

1. Solving cell problem order by order

Cell problem in strong form

$$\operatorname{div}_y \sigma = \eta (\langle \operatorname{div}_x \sigma \rangle - \operatorname{div}_x \sigma) \quad \text{with} \quad \sigma = \mathbf{c} : (\mathbf{E} + (\eta \nabla_x + \nabla_y) \mathbf{Y})$$

Plugging-in series expansions $\mathbf{E} = \sum_k \eta^k \mathbf{E}_k(\mathbf{x})$, $\mathbf{Y} = \sum_k \eta^k \mathbf{Y}_k$ yields

$$\begin{array}{lll} \operatorname{div}_y \sigma_0 = \mathbf{0} & \text{with} & \sigma_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_y \mathbf{Y}_0) \\ \operatorname{div}_y \sigma_1 = \langle \operatorname{div}_x \sigma_0 \rangle - \operatorname{div}_x \sigma_0 & \text{with} & \sigma_1 = \mathbf{c} : (\mathbf{E}_1 + \nabla_x \mathbf{Y}_0 + \nabla_y \mathbf{Y}_1) \quad \mathbf{Y}_0 = \Psi_0 : \mathbf{E}_0, \sigma_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_y \mathbf{Y}_0) \\ \operatorname{div}_y \sigma_2 = \langle \operatorname{div}_x \sigma_1 \rangle - \operatorname{div}_x \sigma_1 & \text{with} & \sigma_2 = \mathbf{c} : (\mathbf{E}_2 + \nabla_x \mathbf{Y}_1 + \nabla_y \mathbf{Y}_2) \quad \mathbf{Y}_1 = \Psi_0 : \mathbf{E}_1 + \Psi_1 \cdot \nabla \mathbf{E}_0, \\ & & \sigma_1 = \mathbf{c} : (\mathbf{E}_1 + \nabla_x \mathbf{Y}_0 + \nabla_y \mathbf{Y}_1) \end{array}$$

solution writes

$$\begin{aligned} \mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = & \Psi_0(\mathbf{y}) : \mathbf{E}_0(\mathbf{x}) + \eta (\Psi_0(\mathbf{y}) : \mathbf{E}_1(\mathbf{x}) + \Psi_1(\mathbf{y}) \cdot \nabla \mathbf{E}_0(\mathbf{x})) \\ & + \eta^2 (\Psi_0(\mathbf{y}) : \mathbf{E}_2(\mathbf{x}) + \Psi_1(\mathbf{y}) \cdot \nabla \mathbf{E}_1(\mathbf{x}) + \Psi_2(\mathbf{y}) \cdot \nabla^2 \mathbf{E}_0(\mathbf{x})) + \mathcal{O}(\eta^3) \end{aligned}$$

1. Solving cell problem order by order

Cell problem in strong form

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma} = \eta (\langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} \rangle - \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}) \quad \text{with} \quad \boldsymbol{\sigma} = \mathbf{c} : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y})$$

Plugging-in series expansions $\mathbf{E} = \sum_k \eta^k \mathbf{E}_k(\mathbf{x})$, $\mathbf{Y} = \sum_k \eta^k \mathbf{Y}_k$ yields

$$\begin{array}{lll} \operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_0 = \mathbf{0} & \text{with} & \boldsymbol{\sigma}_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_{\mathbf{y}} \mathbf{Y}_0) \\ \operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_1 = \langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_0 \rangle - \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_0 & \text{with} & \boldsymbol{\sigma}_1 = \mathbf{c} : (\mathbf{E}_1 + \nabla_{\mathbf{x}} \mathbf{Y}_0 + \nabla_{\mathbf{y}} \mathbf{Y}_1) \quad \mathbf{Y}_0 = \boldsymbol{\Psi}_0 : \mathbf{E}_0, \boldsymbol{\sigma}_0 = \mathbf{c} : (\mathbf{E}_0 + \nabla_{\mathbf{y}} \mathbf{Y}_0) \\ \operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_2 = \langle \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_1 \rangle - \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_1 & \text{with} & \boldsymbol{\sigma}_2 = \mathbf{c} : (\mathbf{E}_2 + \nabla_{\mathbf{x}} \mathbf{Y}_1 + \nabla_{\mathbf{y}} \mathbf{Y}_2) \quad \mathbf{Y}_1 = \boldsymbol{\Psi}_0 : \mathbf{E}_1 + \boldsymbol{\Psi}_1 \cdot \nabla \mathbf{E}_0, \\ & & \boldsymbol{\sigma}_1 = \mathbf{c} : (\mathbf{E}_1 + \nabla_{\mathbf{x}} \mathbf{Y}_0 + \nabla_{\mathbf{y}} \mathbf{Y}_1) \end{array}$$

solution writes

$$\begin{aligned} \mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = & \boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}_0(\mathbf{x}) + \eta (\boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}_1(\mathbf{x}) + \boldsymbol{\Psi}_1(\mathbf{y}) \cdot \nabla \mathbf{E}_0(\mathbf{x})) \\ & + \eta^2 (\boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}_2(\mathbf{x}) + \boldsymbol{\Psi}_1(\mathbf{y}) \cdot \nabla \mathbf{E}_1(\mathbf{x}) + \boldsymbol{\Psi}_2(\mathbf{y}) \cdot \nabla^2 \mathbf{E}_0(\mathbf{x})) + \mathcal{O}(\eta^3) \end{aligned}$$

can be factorized using $\mathbf{E} = \mathbf{E}_0 + \eta \mathbf{E}_1 + \eta^2 \mathbf{E}_2$

$$\mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = \boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}(\mathbf{x}) + \eta \boldsymbol{\Psi}_1(\mathbf{y}) \cdot \nabla \mathbf{E}(\mathbf{x}) + \eta^2 \boldsymbol{\Psi}_2(\mathbf{y}) \cdot \nabla^2 \mathbf{E}(\mathbf{x}) + \mathcal{O}(\eta^3)$$

By-passing variational approach, remember strong form of effective problem

$$\operatorname{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}^* \rangle (\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad \text{on } V,$$

with

$$\begin{aligned} \langle \boldsymbol{\sigma}^* \rangle &= \langle \mathbf{c} : (\mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^*) \rangle = \mathbf{P}_0^* : \mathbf{E}_0 \\ &\quad + \eta (\mathbf{P}_0^* : \mathbf{E}_1 + \mathbf{P}_1^* : \nabla \mathbf{E}_0) \\ &\quad + \eta^2 (\mathbf{P}_0^* : \mathbf{E}_2 + \mathbf{P}_1^* : \nabla \mathbf{E}_1 + \mathbf{P}_2^* : \nabla^2 \mathbf{E}_0) + \mathcal{O}(\eta^3) \end{aligned}$$

yields cascade effective local problems with non-local source terms

$$\operatorname{div}_{\mathbf{x}} \mathbf{P}_0^* : \mathbf{E}_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0} \quad \text{solve for } \mathbf{U}_0$$

$$\operatorname{div}_{\mathbf{x}} (\mathbf{P}_0^* : \mathbf{E}_1(\mathbf{x}) + \mathbf{P}_1^* : \nabla \mathbf{E}_0(\mathbf{x})) = 0 \quad \text{solve for } \mathbf{U}_1$$

$$\operatorname{div}_{\mathbf{x}} (\mathbf{P}_0^* : \mathbf{E}_2(\mathbf{x}) + \mathbf{P}_1^* : \nabla \mathbf{E}_1(\mathbf{x}) + \mathbf{P}_2^* : \nabla^2 \mathbf{E}_0(\mathbf{x})) = 0 \quad \text{solve for } \mathbf{U}_2$$

By-passing variational approach, remember strong form of effective problem

$$\operatorname{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}^* \rangle(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad \text{on } V,$$

with

$$\begin{aligned} \langle \boldsymbol{\sigma}^* \rangle(\mathbf{x}) = & \mathbf{P}_0^* : \mathbf{E}_0(\mathbf{x}) \\ & + \eta (\mathbf{P}_0^* : \mathbf{E}_1(\mathbf{x}) + \mathbf{P}_1^* :: \nabla \mathbf{E}_0(\mathbf{x})) \\ & + \eta^2 (\mathbf{P}_0^* : \mathbf{E}_2(\mathbf{x}) + \mathbf{P}_1^* :: \nabla \mathbf{E}_1(\mathbf{x}) + \mathbf{P}_2^* :: \nabla^2 \mathbf{E}_0(\mathbf{x})) + \mathcal{O}(\eta^3) \end{aligned}$$

Summing-up effective equations: *criminal Ansatz* [Bakhvalov & Panasenko, '84] [Allaire et al 2018]

$$\operatorname{div}_{\mathbf{x}} (\mathbf{P}_0^* : \mathbf{E}(\mathbf{x}) + \mathbf{P}_1^* :: \nabla \mathbf{E}(\mathbf{x}) + \mathbf{P}_2^* :: \nabla^2 \mathbf{E}(\mathbf{x})) + \mathbf{f}(\mathbf{x}) = 0 \quad \text{solve for } \mathbf{U} = \mathbf{U}_0 + \eta \mathbf{U}_1 + \eta^2 \mathbf{U}_2$$

2. Back to variational view: effective energy

Adopting the *criminal* view ($\mathbf{E} = \mathbf{E}_0 + \eta \mathbf{E}_1 + \eta^2 \mathbf{E}_2$)

$$\mathbf{Y}^*(\mathbf{x}, \mathbf{y}) = \boldsymbol{\Psi}_0(\mathbf{y}) : \mathbf{E}(\mathbf{x}) + \eta \boldsymbol{\Psi}_1(\mathbf{y}) \cdot \nabla \mathbf{E}(\mathbf{x}) + \eta^2 \boldsymbol{\Psi}_2(\mathbf{y}) :: \nabla \mathbf{E}(\mathbf{x}) + \mathcal{O}(\eta^3)$$

microscopic strain writes

$$\boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) = \mathbf{E} + (\eta \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}) \mathbf{Y}^* = \mathbf{F}_0(\mathbf{y}) : \mathbf{E}(\mathbf{x}) + \eta \mathbf{F}_1(\mathbf{y}) \cdot \nabla \mathbf{E}(\mathbf{x}) + \eta^2 \mathbf{F}_2(\mathbf{y}) :: \nabla^2 \mathbf{E}(\mathbf{x}) + \mathcal{O}(\eta^3)$$

Effective energy

$$\Phi^*[U] = \int_{V_{\text{int}}} (\langle \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) : \mathbf{c}(\mathbf{y}) : \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) \rangle - \mathbf{f} U) \, d\mathbf{x}$$

2. Back to variational view: effective energy

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Effective energy

$$\Phi^*[U] = \int_{V_{\text{int}}} (\langle (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 :: \nabla^2 \mathbf{E}) : \mathbf{c}(\mathbf{y}) : (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 :: \nabla^2 \mathbf{E}) \rangle - \mathbf{f} \cdot \mathbf{U}) \, d\mathbf{x}$$

Expanding + truncating the effective energy

$$\Phi^*[U] = \frac{1}{2} \int_{V_{\text{int}}} \mathbf{E} : \mathbf{K} : \mathbf{E} \, d\mathbf{x}$$

$$- \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) \, d\mathbf{x}$$

with $\mathbf{K} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_0 \rangle$

2. Back to variational view: effective energy

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Effective energy

$$\Phi^*[U] = \int_V \langle (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 \div \nabla^2 \mathbf{E}) : \mathbf{c}(\mathbf{y}) : (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 \div \nabla^2 \mathbf{E}) \rangle - \mathbf{f} \cdot \mathbf{U} \, d\mathbf{x}$$

Expanding + truncating the effective energy

$$\Phi^*[U] = \frac{1}{2} \int_{V_{\text{int}}} \mathbf{E} : \mathbf{K} : \mathbf{E} \, d\mathbf{x} + \frac{1}{2} \int_{V_{\text{int}}} \eta (\mathbf{E} : \mathbf{A} \div \nabla \mathbf{E} + \nabla \mathbf{E} \div \mathbf{A}^T : \mathbf{E}) \, d\mathbf{x} - \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) \, d\mathbf{x}$$

$$\text{with } \mathbf{K} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_0 \rangle \quad \mathbf{A} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_1 \rangle$$

2. Back to variational view: effective energy

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Effective energy

$$\Phi^*[U] = \int_V \langle (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 \div \nabla^2 \mathbf{E}) : \mathbf{c}(\mathbf{y}) : (\mathbf{F}_0 : \mathbf{E} + \eta \mathbf{F}_1 \div \nabla \mathbf{E} + \eta^2 \mathbf{F}_2 \div \nabla^2 \mathbf{E}) \rangle - \mathbf{f} \cdot \mathbf{U} \, d\mathbf{x}$$

Expanding + truncating the effective energy

$$\begin{aligned} \Phi^*[U] &= \frac{1}{2} \int_{V_{\text{int}}} \mathbf{E} : \mathbf{K} : \mathbf{E} \, d\mathbf{x} + \frac{1}{2} \int_{V_{\text{int}}} \eta (\mathbf{E} : \mathbf{A} \div \nabla \mathbf{E} + \nabla \mathbf{E} \div \mathbf{A}^T : \mathbf{E}) \, d\mathbf{x} \\ &\quad + \frac{1}{2} \int_{V_{\text{int}}} \eta^2 (\nabla \mathbf{E} \div \mathbf{B} \div \nabla \mathbf{E} + \mathbf{E} : \mathbf{C} \div \nabla^2 \mathbf{E} + \nabla^2 \mathbf{E} \div \mathbf{C}^T : \mathbf{E}) \, d\mathbf{x} - \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) \, d\mathbf{x} + \mathcal{O}(\eta^3) \end{aligned}$$

$$\text{with } \mathbf{K} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_0 \rangle \quad \mathbf{A} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_1 \rangle \quad \mathbf{B} = \langle \mathbf{F}_1^T : \mathbf{c} : \mathbf{F}_1 \rangle \quad \mathbf{C} = \langle \mathbf{F}_0^T : \mathbf{c} : \mathbf{F}_2 \rangle$$

3. Back to variational view: equilibrium

Stationarity condition for the energy writes

$$\int_{V_{\text{int}}} [\mathbf{E} : \mathbf{K} : \delta \mathbf{E} + \eta (\mathbf{E} : \mathbf{A} :: \nabla \delta \mathbf{E} + \nabla \mathbf{E} :: \mathbf{A}^T : \delta \mathbf{E})] dx + \int_{V_{\text{int}}} \eta^2 [\mathbf{E} :: \mathbf{B} :: \nabla \delta \mathbf{E} + \mathbf{E} : \mathbf{C} :: \nabla^2 \delta \mathbf{E} + \nabla^2 \mathbf{E} :: \mathbf{C}^T : \delta \mathbf{E}] dx - \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \delta \mathbf{U}(\mathbf{x}) dx = 0$$

3. Back to variational view: equilibrium

Stationarity condition for the energy writes

$$\int_{V_{\text{int}}} \left[\underbrace{\mathbf{E} : \mathbf{K} : \delta \mathbf{E}}_{=(\mathbf{P}_0^* : \mathbf{E}) : \delta \mathbf{E}} + \eta \left(\underbrace{\mathbf{E} : \mathbf{A} :: \nabla \delta \mathbf{E} + \nabla \mathbf{E} :: \mathbf{A}^T : \delta \mathbf{E}}_{=(\mathbf{P}_1^* :: \nabla \mathbf{E}) : \delta \mathbf{E}} \right) \right] dx$$

$$+ \int_{V_{\text{int}}} \eta^2 \left[\underbrace{\nabla \mathbf{E} :: \mathbf{B} :: \nabla \delta \mathbf{E} + \mathbf{E} : \mathbf{C} :: \nabla^2 \delta \mathbf{E} + \nabla^2 \mathbf{E} :: \mathbf{C}^T : \delta \mathbf{E}}_{=(\mathbf{P}_2^* :: \nabla^2 \mathbf{E}) : \delta \mathbf{E}} \right] dx - \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) dx = 0$$

equivalent to

$$\text{div}_{\mathbf{x}} (\mathbf{P}_0^* : \mathbf{E}(\mathbf{x}) + \mathbf{P}_1^* :: \nabla \mathbf{E}(\mathbf{x}) + \mathbf{P}_2^* :: \nabla^2 \mathbf{E}(\mathbf{x})) + \mathbf{f}(\mathbf{x}) = 0$$

3. Back to variational view: equilibrium

Stationarity condition for the energy writes

$$\int_{V_{\text{int}}} \left[\underbrace{\mathbf{E} : \mathbf{K} : \delta \mathbf{E}}_{=(\mathbf{P}_0^* : \mathbf{E}) : \delta \mathbf{E}} + \eta \left(\underbrace{\mathbf{E} : \mathbf{A} : \nabla \delta \mathbf{E} + \nabla \mathbf{E} : \mathbf{A}^T : \delta \mathbf{E}}_{=(\mathbf{P}_1^* : \nabla \mathbf{E}) : \delta \mathbf{E}} \right) \right] dx$$

$$+ \int_{V_{\text{int}}} \eta^2 \left[\underbrace{\nabla \mathbf{E} : \mathbf{B} : \nabla \delta \mathbf{E} + \mathbf{E} : \mathbf{C} : \nabla^2 \delta \mathbf{E} + \nabla^2 \mathbf{E} : \mathbf{C}^T : \delta \mathbf{E}}_{=(\mathbf{P}_2^* : \nabla^2 \mathbf{E}) : \delta \mathbf{E}} \right] dx - \int_{V_{\text{int}}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) dx = 0$$

equivalent to

$$\text{div}_{\mathbf{x}} (\mathbf{P}_0^* : \mathbf{E}(\mathbf{x}) + \mathbf{P}_1^* : \nabla \mathbf{E}(\mathbf{x}) + \mathbf{P}_2^* : \nabla^2 \mathbf{E}(\mathbf{x})) + \mathbf{f}(\mathbf{x}) = 0$$

gradient term writes $(\mathbf{P}_2^* : \nabla^2 \mathbf{E}) : \delta \mathbf{E} = (-\tilde{\mathbf{B}} : \nabla^2 \mathbf{E}) : \delta \mathbf{E}$ with $\tilde{\mathbf{B}} = \mathbf{B} - \mathbf{C} - \mathbf{C}^T$

In many cases [Smyshlyaev+Cherednichenko, 2000] [Allaire, 2016, 2018] [Le+Marigo, 2018]

$$\tilde{\mathbf{B}} < 0 \quad \text{“incorrect sign”}$$

→ challenge when solving associated BVP: creates oscillating terms in the solution

- Asymptotic homogenization for linear elastostatics
- **A sign problem**
- Is it worth the effort?

Origin of the sign problem

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Where was positivity lost in the homogenization process? Recall, effective energy writes (with $\mathbf{f} = 0$)

$$\Phi^*[U] = \int \frac{1}{2} \langle \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) : \mathbf{c}(\mathbf{y}) : \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) \rangle d\mathbf{x} \geq 0$$

with

$$\boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) = \mathbf{F}_0(\mathbf{y}) : \mathbf{E}(\mathbf{x}) + \eta \mathbf{F}_1(\mathbf{y}) :: \nabla \mathbf{E}(\mathbf{x}) + \eta^2 \mathbf{F}_2(\mathbf{y}) :: \nabla^2 \mathbf{E}(\mathbf{x}) + \mathcal{O}(\eta^3)$$

Origin of the sign problem

Where was positivity lost in the homogenization process? Recall, effective energy writes (with $\mathbf{f} = 0$)

$$\Phi^*[U] = \int \frac{1}{2} \langle \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) : \mathbf{c}(\mathbf{y}) : \boldsymbol{\varepsilon}^*(\mathbf{x}, \mathbf{y}) \rangle d\mathbf{x} \geq 0$$

This writes, in matrix notation

$$\Phi^*[U] = \int \frac{1}{2} \begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \eta \nabla \mathbf{E}(\mathbf{x}) \\ \eta^2 \nabla^2 \mathbf{E}(\mathbf{x}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{K} & \mathbf{A} & \mathbf{C} \\ \mathbf{A} & \mathbf{B} & \dots \\ \mathbf{C}^T & \dots & \dots \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \eta \nabla \mathbf{E}(\mathbf{x}) \\ \eta^2 \nabla^2 \mathbf{E}(\mathbf{x}) \end{pmatrix} d\mathbf{x} + \mathcal{O}(\eta^3)$$

with

$$\mathbf{K} > 0, \mathbf{B} \geq 0 \text{ and } \begin{pmatrix} \mathbf{K} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{B} \end{pmatrix} \geq 0$$

Truncation + integration by parts has led to

$$(\mathbf{P}_2^* :: \nabla^2 \mathbf{E}) : \delta \mathbf{E} = (-\tilde{\mathbf{B}} :: \nabla^2 \mathbf{E}) : \delta \mathbf{E} \quad \text{with} \quad \tilde{\mathbf{B}} = \mathbf{B} - \mathbf{C} - \mathbf{C}^T < 0$$

→ Can we change truncation rule to preserve positivity?

Minimal truncation preserving positivity

using block-Cholesky decomposition $\mathbf{L} \cdot \mathbf{D} \cdot \mathbf{L}^T$

[Thbaut+Audoly+CL, 2025]

$$\begin{pmatrix} \mathbf{K} & \mathbf{A} & \mathbf{C} & \cdots \\ \mathbf{A}^T & \mathbf{B} & \cdots & \\ \mathbf{C}^T & \cdots & & \\ \cdots & & & \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{L}_1 & \mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{L}_2 & \cdots & & \\ \cdots & & & \end{pmatrix} \cdot \begin{pmatrix} \mathbf{D}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{D}_1 & \cdots & \\ \mathbf{0} & \cdots & & \\ \cdots & & & \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{L}_1 & \mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{L}_2 & \cdots & & \\ \cdots & & & \end{pmatrix}^T$$

identifying order by order yields

iteration order	0	1	2
moduli	$\mathbf{D}_0 = \mathbf{K}$		$\mathbf{D}_1 = \mathbf{B} - \mathbf{A}^T \cdot \mathbf{K}^{-T} \cdot \mathbf{A}$
coefficients		$\mathbf{L}_1 = (\mathbf{K}^{-1} \cdot \mathbf{A})^T$	$\mathbf{L}_2 = (\mathbf{K}^{-1} \cdot \mathbf{C})^T$

- with “correct” signs, $\mathbf{D}_0 > 0$, $\mathbf{D}_1 \geq 0$, positivity of \mathbf{D}_1 follows from $\begin{pmatrix} \mathbf{K} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{B} \end{pmatrix} \geq 0$
- applicable up to any order

- Asymptotic homogenization for linear elastostatics
- A sign problem
- **Is it worth the effort?**

Energy formulation, beam network

$$\Phi_{\text{discrete}}(\mathbf{v}_i, \theta_i) = \frac{1}{2} \sum_{\alpha} \left(EA \ell \varepsilon_{\alpha}^2 + \frac{EI}{\ell} (\kappa_{\alpha}^2 + 12 \tau_{\alpha}^2) \right)$$

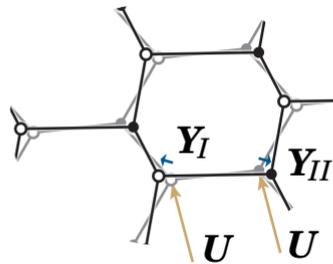
Two-scale Ansatz

$$\begin{pmatrix} \mathbf{v}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{U}(\mathbf{x}) \\ \boldsymbol{\gamma}(\mathbf{x}) \end{pmatrix} + \mathbf{Y}_{b(i)}(\mathbf{x}) \quad \text{with}$$

- **macroscopic** displacement $\mathbf{U}(\mathbf{x})$, strain $\mathbf{E}(\mathbf{x}) = \nabla_{\mathbf{x}} \mathbf{U}(\mathbf{x})$
- **microscopic** displacement + rotation $\mathbf{Y}_b(\mathbf{x}) = (\boldsymbol{\xi}_b(\mathbf{x}), \psi_b(\mathbf{x}))$

use $\sum_{\alpha} \approx \int \dots d\mathbf{x}$: *continualized* strain energy $\Phi[\mathbf{U}, \mathbf{Y}]$

→ Homogenization yields $\mathbf{Y}^*(\mathbf{x}) = \boldsymbol{\Psi}_0 \cdot \mathbf{E}(\mathbf{x}) + \eta \boldsymbol{\Psi}_1 : \nabla \mathbf{E}(\mathbf{x})$



Provided a lattice geometry, definitions of \mathcal{H} , \mathbf{E}_α , the variational homogenization procedure yields

$$\begin{cases} \mathbf{Y}^*(\mathbf{x}) = \Psi_0 \cdot \mathbf{E}(\mathbf{x}) + \eta \Psi_1 : \nabla \mathbf{E}(\mathbf{x}) \\ \Phi^*[\mathbf{U}] = \frac{1}{2} \int [\mathbf{E} \cdot \mathbf{K} \cdot \mathbf{E} + \eta \mathbf{E} \cdot \mathbf{A} : \nabla \mathbf{E} + \eta^2 (\mathbf{B} :: (\nabla \mathbf{E} \otimes \nabla \mathbf{E}) + 2 \mathbf{E} : \mathbf{C} :: \nabla^2 \mathbf{E})] dx \end{cases}$$

Second-order HOMogenization Automated in a Library (SHOAL) [Audoly, 2023]

- generic, higher-order homogenization for linear, discrete elastic structures
- written in Wolfram Mathematica
- explicit, algebraic calculations
- allows for varying geometric & elastic properties
- based on [Audoly+CL, 2023]



based on post-processing discrete simulations (**no solution of effective BVP**)

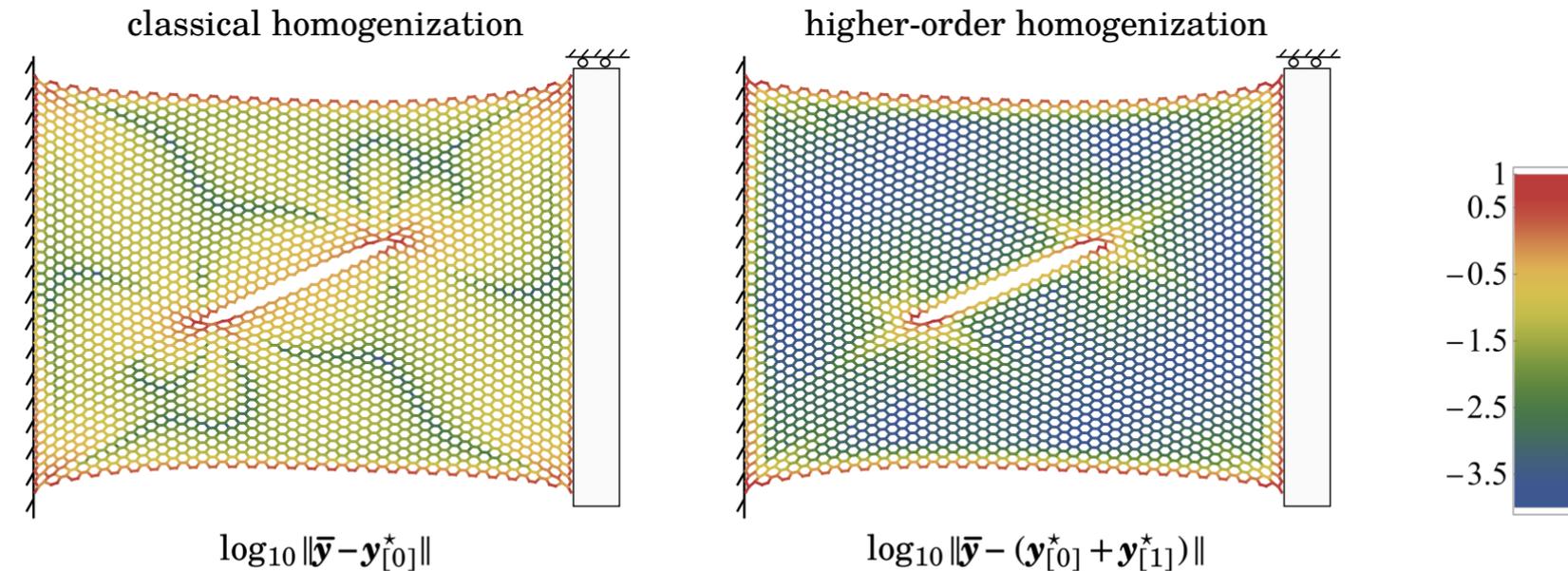
Given a beam in the discrete lattice, compute

- smooth interpolations $\bar{\mathbf{v}}_i(\mathbf{x})$ for all Bravais sublattices $1 \leq i \leq n_b$
- macroscopic displacement $\bar{\mathbf{U}} = \langle \bar{\mathbf{v}}_i \rangle_{1 \leq i \leq n_b}$, $\bar{\mathbf{E}} = \nabla \bar{\mathbf{U}}$ and $\nabla \bar{\mathbf{E}}$
- microscopic displacement $\bar{\mathbf{y}} = (\bar{\mathbf{v}}_i - \bar{\mathbf{U}})_{1 \leq i \leq n_b}$
- **compare with** prediction $\mathbf{y}_{[0]}^* = \Psi_0 \cdot \bar{\mathbf{E}}$ and $\mathbf{y}_{[1]}^* = \Psi_1 : \nabla \bar{\mathbf{E}}$ of homogenized model
→ $\mathbf{e}_{[0]} = \|\bar{\mathbf{y}} - \mathbf{y}_{[0]}^*\|$ and $\mathbf{e}_{[1]} = \|\bar{\mathbf{y}} - (\mathbf{y}_{[0]}^* + \mathbf{y}_{[1]}^*)\|$

Application: crack in honeycomb

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- Validation procedure based on post-processing of discrete simulations
- **this is not** a comparison against predictions obtained from effective models

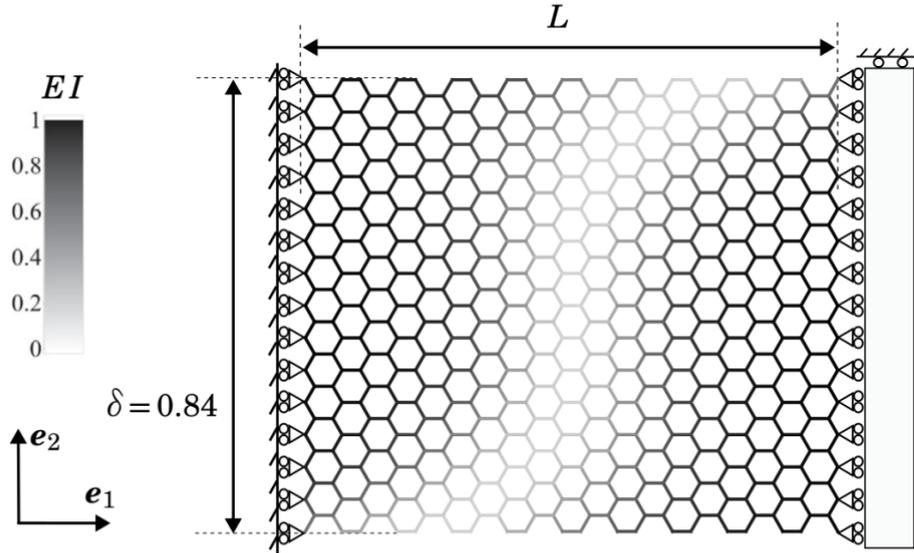


[Ye+CL+Audoly 2024]

Application: varying elastic properties

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- $\nabla(EI)$ enters the gradient prediction



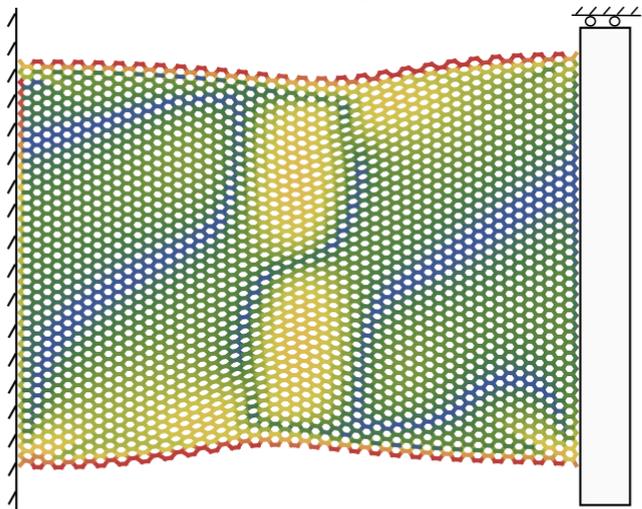
[Ye+CL+Audoly 2024]

Application: varying elastic properties

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- $\nabla(EI)$ enters the gradient prediction

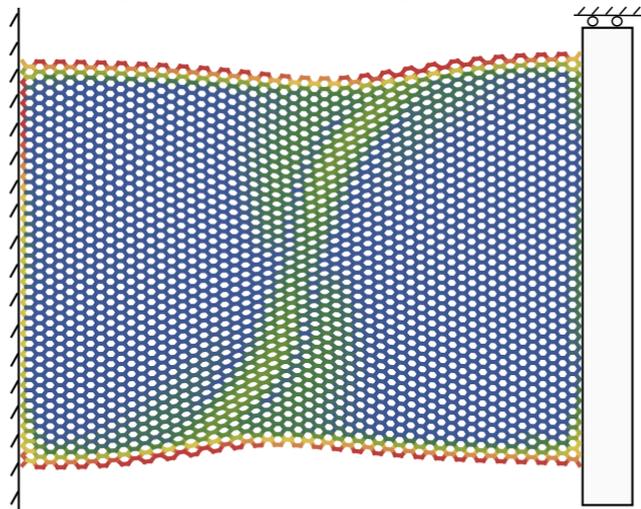
classical homogenization



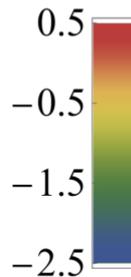
$\log_{10} \|\bar{\mathbf{y}} - \mathbf{y}_{[0]}^*\|$

[Ye+CL+Audoly 2024]

higher-order homogenization



$\log_{10} \|\bar{\mathbf{y}} - (\mathbf{y}_{[0]}^* + \mathbf{y}_{[1]}^*)\|$

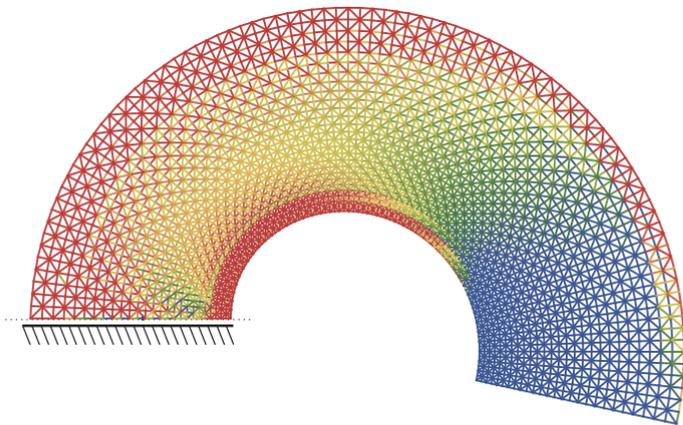


Application: varying geometric properties

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- Circular arch subject to a weight-like force

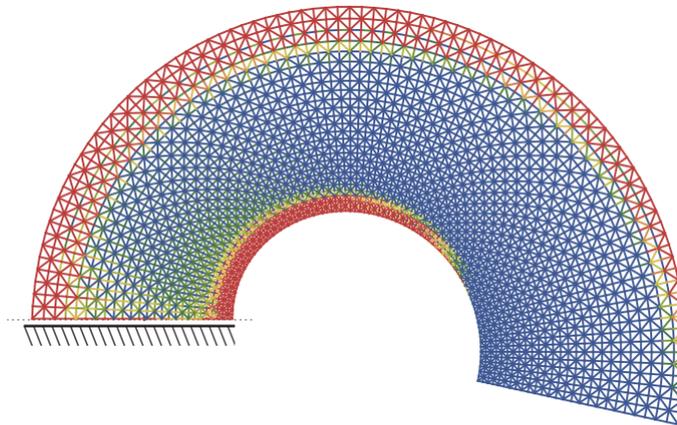
classical homogenization



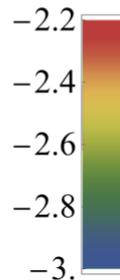
$$\log_{10} \|\bar{\mathbf{y}} - \mathbf{y}_{[0]}^*\|$$

[Ye+CL+Audoly 2024]

higher-order homogenization



$$\log_{10} \|\bar{\mathbf{y}} - (\mathbf{y}_{[0]}^* + \mathbf{y}_{[1]}^*)\|$$



When modeling *elastic* periodic materials

- Asymptotic homogenization
 - provides a rigorous derivation of gradient model
 - accurately captures higher-order effects far from boundaries
 - is not a proof of convergence, but can help identify the limit
- Variational approach + criminal Ansatz
 - ultimately equivalent to strong form, perturbative approach
 - delivers higher-order model in variational form
 - provides tools to overcome sign issues

For discrete microstructures

- cell problem is algebraic
- all calculations can be done automatically in mathematica, download SHOAL

Can we make predictions (ie solve BVPs)?

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The devil is at the boundaries

- limited accuracy in boundary regions (boundary layers)
- higher-order equations require extra BCs
- boundary terms left-out in homogenization procedure

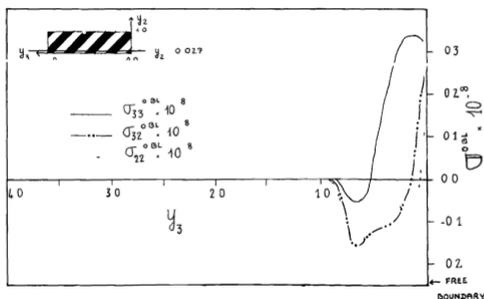
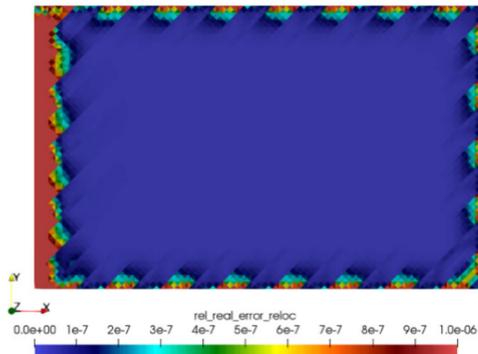


Figure 3. — The boundary layer stresses σ^{obs} plotted against y_3 , at $y_2 = 0.027$ fixed.

[Dumontet, '86]



[Fergoug et al, '22]

How can we handle this?

- analysis of boundary layers + matched expansions [Thbaut+Audoly+CL, 2024]
- energy approach [work in progress, PhD M. Thbaut]

Merci de votre attention !

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